University of California

Santa Barbara

Discrete Representations of the Braid Groups

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy

in

Mathematics

by

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June 2019

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dedicated to Warren H. Scherich Jr.

Acknowledgements

This dissertation is the culmination of 14 years: 6 years of undergrad and 8 years of graduate school. I have had support mathematically, emotionally, financially, physically and spiritually from so many people over the years. I would like to thank just a small subset of those people here.

My advisor, Darren. With out you, this dissertation simply would not exist. Thank you for always being available, both in person and by email. Thank you for helping me to choose interesting projects and teaching me how to be a researcher. I am honored to be one of your students and and to be one little part of your mathematical legacy.

My committee, Stephen and Daryl. Thank you for reading my dissertation and being available for questions along the way.

Ken Millett. Thank you for your endless moral support, not only for my research but for my Math-Dance endeavors. It meant so much to me that you participated in my Math-Dance videos and encouraged me to grow and do it all. I hope to be like you when I grow up.

My father. Well, what can I say, pop? I beat you to it! Thank you for being the rock in my life, always. I am able to take so many more risks and reach so much further because I know I have you to lean back on.

My mother and namesake. Thank you for always being a safe place for me to hide when the world gets too big and scary to handle. I love you so much and cant imagine how I could have survived this PhD without your endless emotional support and validation. I might be a Dr., but I will always be your sweet baby girl.

My Brother, Warren. You will always be my hero. I am so proud of all that you do and have accomplished. I look up to you and want to be just like you. Thank you for inspiring me.

Lindsee Acton. Thank you for being my biggest Math-Dance supporter, and such a loving member of my family. **Brenna Bates-Grub.** Thank you for 25 years of friendship and support. **Amanda Curtis.** Thank you for always being my sanity check, that grads school really is THAT hard. **Michelle Chu.** Thank you for your friendship and mentorship. You have really helped me to feel that I belong in the math community. **Chelsea Cummings.** Thank you for the letting me cry on your shoulder for the last 9 years. **Ruth Deeble.** You were the first person to predict that I would become a doctor one day. How did you know? Rest in peace, grandma. **Zsuzsanna Dancso.** Thank you for taking a chance on me, an unknown mathematician. You are such a wonderful collaborator and mentor. I am so honored to know you and lucky to work with you. **Chris Gorman.** Thank you for always being there. **Dana Iltis.** Thank you for your support and friendship. **Trindel Maine.** You were my first female mathematical role model. Thank you for your support over all these years. **Dean Morales.** Thanks for being there. **Bonnie, John and Teza Ross.** Thank you for continuing to be my safety net. **Warren Scherich Sr.** I wish you could have been here to see me graduate. Thank you for all of your support and love. Rest in peace, grandpa. **Eric Seidman.** Thank you for emotionally supporting me through my qualifying exams. May you rest in peace and I hope you know that you are missed. **Sheri Tamagawa.** Best teammate a girl could ask for. Thanks for being right by my side, every step of the way through this crazy ordeal of grad school. **Steve Trettel.** Thank you for being so darn good at math and helping me whenever I got stuck. **Anush Tserunyan.** Thank you for believing in me as an undergrad and encouraging me to go to grad school.

Thanks to: Kate Hake, Beatrice Martino, Medina Price, Vicky and Dean Morales, Darwin Pawington, Alex Nye, Lauren Breese, Delmer Reed, Kevin Malta, Steve Deeble, Harley Busch, Jeremy Busch, Marte Aponte, Donna Oden, Emily Baker, Abe Pressman, Lana Smith-Hale, Arica Lubin, Wendy Ibsen, Ofelia Aguirre Paden, Liam Watson, Dror Bar-Natan, Bill Bogley, and Jozef Przytycki.

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Real Discrete Specializations of the Burau Representation for B_3, Mathematical Proceedings of the Cambridge Philosophical Society, 1-10. doi:10.1017/S0305004118000683 https://arxiv.org/abs/1801.08203

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Abstract

Discrete Representations of the Braid Groups

by

Nancy Catherine Scherich

Many well known representations of the braid groups are parameterized by a complex parameter, such as the Burau, Jones and BMW representations. This dissertation develops a construction for choosing specializations of the parameters so the images of the representations are discrete groups. This construction requires not only a parameterized representation, but the representations need to be sesquilinear. Squier showed that the Burau representation is sesquilinear. This dissertation extends Squier's result to all of the Jones and BMW representations, and finds discrete specializations of these representations.

Contents

Chapter 1

Introduction

1.1 Overview

Representations of the braid groups have attracted attention because of their wide variety of applications from discrete geometry to quantum computing. Two well studied representations are the Jones representations and one of its irreducible summands, the Burau representation. These representations are parameterized by a variable q (or conventionally t for the Burau representation), and much work has been done to understand the structure of the images for specializations of the parameter, as depicted in Figure 1.1.

For example, the Jones representations of the braid groups collapse to a representation of the symmetric group, Σ_n, when specializing $q = 1$. When $t = -1$, the Burau representation is symplectic and has been studied by Brendle, Margalit and Putman in [6]. The Jones representations are used in modeling quantum computations, so much work has been done to understand specializations to roots of unity, as explored by Funar and Kohno in [12], Freedman, Larson and Wang in [11], and many others. Venkataramana in [30] showed the Burau representation is arithmetic for certain specializations to roots of unity.

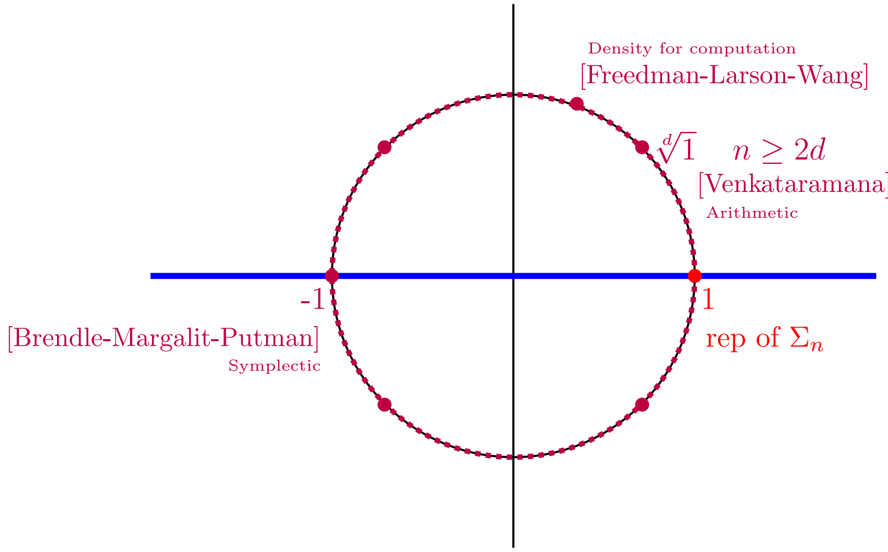


Figure 1.1: Structural results for specializations of the Burau representation.

However, there seems to be a lack of exploration of the *real* specializations of these representations. The main focus of this dissertation is to find real specializations of parameterized representations of the braid groups so that the images are discrete groups.

As a warm up, Chapter 2 focuses only on the Burau representation, and Section 2.2 proves the following complete classification of the real discrete specializations on B_3.

Theorem. *The real discrete specializations of the Burau representation of B_3 are exactly when t satisfies one of the following:*

1. $t < 0$ *and* $t \neq -1$
2. $0 < t \leq \frac{3-\sqrt{5}}{2}$ *or* $t \geq \frac{3+\sqrt{5}}{2}$
3. $\frac{3-\sqrt{5}}{2} < t < \frac{3+\sqrt{5}}{2}$ *and the image forms a triangle group.*

Additionally, the specialization is faithful in (1) *and* (2).

The remainder of the dissertation is dedicated to the proof and application of the following main result.

Main Result. *Let $\rho_q : B_n \to GL_m(\mathbb{Z}[q^{\pm 1}])$ be a braid group representation with a parameter q. Suppose there exists a matrix J_q so that:*

1. *for all M in the image of ρ_q, $M^* J_q M = J_q$, where by definition $M^*(q) = M^\intercal(\frac{1}{q})$,*
2. *$J_q = (J_{\frac{1}{q}})^\intercal$,*
3. *J_q is positive definite for q in a complex neighborhood η of 1.*

Then, there exists infinitely many Salem numbers s, so that the specialization representation ρ_s at $q = s$ is discrete.

This result gives a constructive way to find infinite classes of real specializations at certain algebraic numbers, called Salem numbers, so that that the images of the specialized representations are discrete. Representations satisfying property 1 in the main result are called *sesquilinear*, or sometimes *unitary*. The sesquilinearity property can be described by saying the image of the representation is a subset of a generalized unitary group. The discreteness is more of a property about the target unitary groups than of the braid groups. So really this theorem applies to sesquilinear representations of *any* group, not just the braid groups. A generalized statement of the main result is proved in detail in Chapter 3, as well as a discussion of sesquilinear representations and generalized unitary groups.

The next hurdle is to find representations that are in fact sesquilinear. Squier showed in [28] that the Burau representation is sesquilinear and satisfies the criteria for the main result. Chapters 4 and 5 are dedicated to extending Squier's result to all of the Jones representations and the BMW representations of the braid groups. Since these representations are sesquilinear, then the main result applies and specific examples of discrete specializations of the Jones and BMW representations are computed.

Discreteness is an interesting structural property to study in light of the current pursuit of thin groups and lattices. It turns out that the images of the braid group representations in the main result are subgroups of lattices inside $GL_n(\mathbb{R})$. Chapter 6 explores the lattice structure and some commensurability results of the target lattices.

1.2 The Braid Groups

The braid groups are a very exciting and versatile mathematical object that are interesting from an algebraic, geometric and topological point of view. The group presentation given below was first introduced by E. Artin in 1925 [1].

Definition 1.2.1. *The braid group on n strands, denoted B_n, is a group with the following presentation:*

Generators: $\sigma_1, \cdots, \sigma_{n-1}$

Relations: $\sigma_i\sigma_j = \sigma_j\sigma_i$ *for* $|i-j| > 1$ *(far commutativity)*

$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ *for all* i *(braid relation)*

From this algebraic perspective, we can easily see that this group is finitely generated, finitely presented and infinite as each generator has infinite order. Also, the braid relation can be rearranged

$$(\sigma_{i+1}\sigma_i)^{-1}\sigma_i(\sigma_{i+1}\sigma_i) = \sigma_{i+1}$$

to show that the generators are conjugate. This is a particularly useful fact when studying representations of the braid groups.

What is difficult to see from this algebraic definition is the motivation for the two sets of relations. Viewing the braids from a more geometric perspective helps to see this motivation.

Braids in B_n can be described as diagrams with n strands, which are stacks of the generating diagrams σ_i and σ_i^{-1} defined in Figure 1.2.

Figure 1.2: Generating diagrams for the braid group.

In σ_i, the strand in the i'th position crosses downwards behind the strand in the $i+1$ position, and in σ_i^{-1} the strand in the i'th position cross downwards in front of the strand in the $i+1$'th position. The braid group on n-strands is the collection of diagrams created by stacking the σ_i's, considered up to a certain isotopy of the strands. Each braid can be described by listing the σ_i's that occur in order from bottom to top. The group multiplication is visualized by diagram stacking.

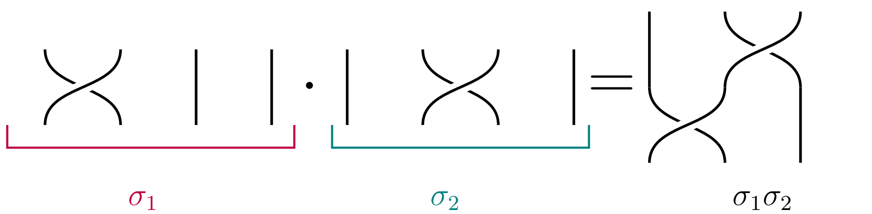


Figure 1.3: Multiplication is diagram stacking.

Importantly, what distinguishes a braid from a more general tangle is the monotonicity of the strands, and the crossings occur at distinct heights in the braid. This is best seen by orienting the strands with an upward flow. Braids are only considered up to isotopy of the strands relative to the endpoints and which preserves the monotonicity of the strands.

Example: The tangle in Figure 1.4 is not a braid because it can not be isotoped relative the endpoints so that the strands flow monotonically upwards.

Example: The tangle in Figure 1.5 is a braid because it can be isotoped relative endpoints to the braid $(\sigma_1^{-1})^3$.

The far commutativity relation is easy to visualize with this geometric perspective.

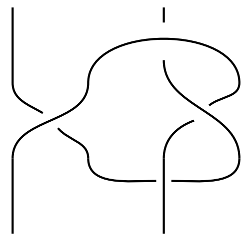


Figure 1.4: A tangle that *is not* a braid.

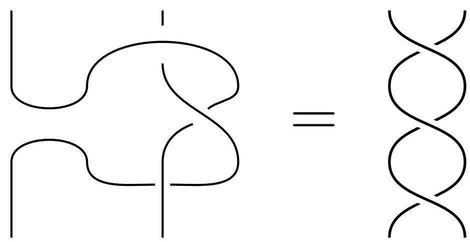
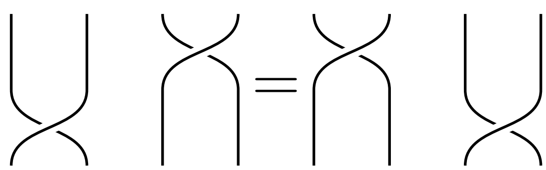
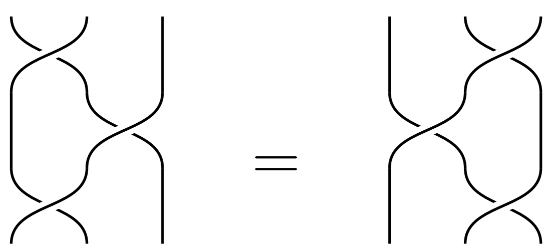


Figure 1.5: A tangle that *is* a braid.



$\sigma_1\sigma_3$ $\sigma_3\sigma_1$

Figure 1.6: Far commutativity relation.



$\sigma_1\sigma_2\sigma_1$ $\sigma_2\sigma_1\sigma_2$

Figure 1.7: The braid relation.

If σ_i and σ_j use disjoint strands in their crossings, then there is an acceptable isotopy that slides the crossings passed each other, as shown in Figure 1.6.

Knot theoretically, the braid relation is easy to see as a Reidemeister III move applied to the strands. Or rather, the middle strand can slide in between the other two strands, which changes the order of the crossings, as shown in Figure 1.7.

The braid relation shows that σ_i and σ_{i+1} do not commute with each other, but rather entangle with each other. Since the generators do not all commute, it is a bit surprising is that the braid groups have a non trivial center.

Theorem 1.2.2 (van Buskirk [29])**.** *The center of B_n is cyclic generated by*

$$(\sigma_1\sigma_2\cdots\sigma_{n-1})^n.$$

Using this visual description, it is easy to visualize that $(\sigma_1\sigma_2\cdots\sigma_{n-1})^n$ is central, but difficult to see that it generates all of the center.

1.3 Where do the braid groups arise in real life?

E. Artin in 1925 [1] was the first person to name the braid groups and give an explicit algebraic presentation for these groups. While this is the most famous introduction of the braid groups, their existence and some deep properties were known far before 1925 in the early descriptions of mapping class groups, by Hurwitz and Fricke-Klein in the late 1800's though these references are difficult to find today.

This section briefly outlines several ways the braid groups arise in various different mathematical settings.

Mapping Class Group

The mapping class group of a topological space M is the group of isotopy classes of homeomorphisms of M. The braid group is the mapping class group of an n-punctured disc where the homeomorphisms fix the boundary. This can be seen by visualizing each puncture connected to the boundary by a string. After the homeomorphism is applied, the punctures have swapped places and the strings are braided.

Fundamental group of configuration space

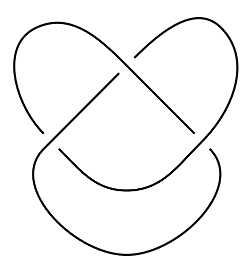
The configuration space of n points is defined to be

$$C_n = \{(x_1, \cdots, x_n) \in \mathbb{C}^n | x_i \neq x_j \text{ if } i \neq j\}.$$

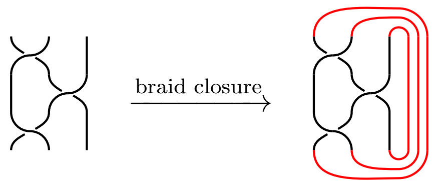
There is a natural action of the symmetric group S_n on C_n by permuting the coordinates. Then B_n is the fundamental group of C_n modulo this action, $B_n \cong \pi_1(C_n)/S_n$.

Knot Theory

A knot is a smooth embedding of the circle S^1 into $\mathbb{R}^3$. A link is an embedding of multiple circles. The knot type of a knot(or link) is the equivalence class of the knot up to ambient isotopy. The major question in knot theory is to determine the knot type of a knot, or tell when two knots are "the same" or "not the same". Knots are often drawn as planar projections with the crossings indicated by a gap in the under strand.



These projections are called knot diagrams. The same knot can have wildly different diagrams, which are related by Reidemeister moves. Braids can serve as one way to standardize these diagrams. Every braid gives rise to a knot or link by taking the braid closure. The braid closure is formed by adding arcs that connect the i'th strand at the top of the braid to the i'th strand at the bottom of the braid.



Theorem 1.3.1 (Alexander's Theorem)**.** *Every knot and link can be realized as the closure of a braid.*

There are several knot invariants that are algorithmically defined by first converting the knot to a closure of a braid. For example, in Chapter 2, the Alexander polynomial of a knot can be computed by first converting the knot to a braid closure and then take the determinant of an adjusted Burau representation of the braid.

Yang-Baxter Equation

The Yang-Baxter equation was originally introduced in the field of statistical mechanics in the late 1960's, and more modernly is closely related to the study of bialgebras. Let V be a finite dimensional vector space and R a linear map on $V \otimes V$. R is said to satisfy the Yang-Baxter equation if

$$(id \otimes R) \circ (R \otimes id) \circ (id \otimes R) = (R \otimes id) \circ (id \otimes R) \circ (R \otimes id) \in End(V^{\otimes 3}),$$

where id is the identity map on V. The Yang-Baxter equation is reminiscent of the braid relation. Invertible solutions to this equation give rise to representations of the Braid Group via $\rho(\sigma_i) = I^{\otimes(i-1)} \otimes R \otimes I^{\otimes(n-i-1)}$.

Quantum Computations

In the 1980's, many models of quantum computation first appeared. In 1997, Kitaev in [17] introduced the idea of a topological quantum computer. A logic gate in a topological quantum computer is a collection of paths taken by anyons, which are two dimensional quasiparticles. For physical and stability reasons, these paths form braids. Braidings of anyons in a topological quantum computer change the encoded quantum information, giving rise to a quantum computation. So, representations of the braid groups can be used to describe quantum computations. [10]

1.4 Representation Theory

This section will define the standard terminology of representation theory that will be used throughout the thesis.

Definition 1.4.1. *A* ***representation*** *of a group G is a group homomorphism $\rho : G \to GL(V)$ for some vector space V. A representation can also be defined in terms of a group action or a module structure.*

Definition 1.4.2. *A representation is* ***irreducible*** *if it has no proper sub-representations. (Under nice circumstances, this is equivalent to a representation which is not a direct sum of representations.)*

Definition 1.4.3. *A representation is* ***faithful*** *if it is an injective homomorphism.*

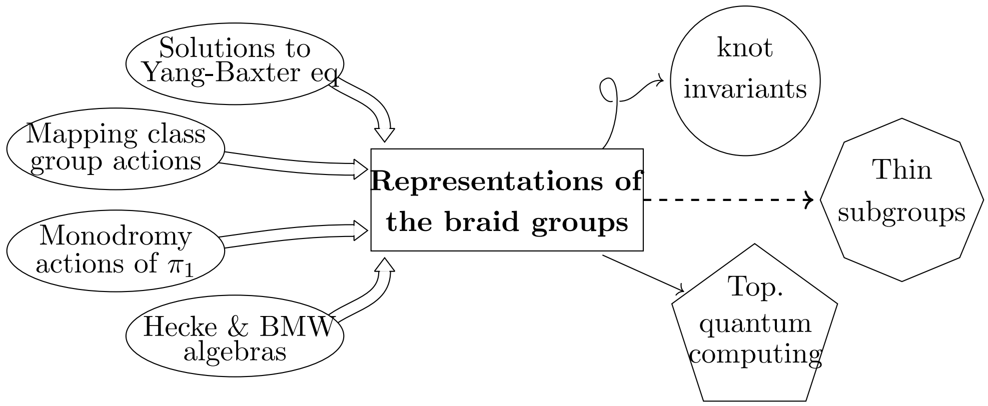
Definition 1.4.4. *A representation is* ***discrete*** *if its image is a discrete subgroup of $GL_m(\mathbb{R})$, with the standard euclidean topology.*

Definition 1.4.5. *A representation is* ***parameterized by a variable*** *t if the image lies in $GL(\mathbb{Z}[t^{\pm 1}])$.*

Definition 1.4.6. *For a parameterized representation ρ, a* ***specialization*** *of ρ at $s \in \mathbb{C}$ is a composition of ρ and the evaluation map $t = s$.*

1.4.1 Representations of the Braid Groups

As described in Section 1.3, the braid groups arise in several different mathematical settings, many of which induce representations of the braid groups.



The Jones representations and one of its irreducible summands, the Burau representation, are very well known representations and are described in detail in the later chapters. These representations are very important for a myriad of reasons, but particularly for the following two properties.

1. The Jones representations parameterize all of the irreducible representations of the braid groups with two eigenvalues.

2. For $n = 4$, Bigelow conjectured and Tetsuya Ito proved that the faithfulness of the Burau representation implies that the Jones polynomial detects the unknot [3, 13]. More precisely, the Burau representation for $n = 4$ is unfaithful if and only if there is a knot with braid index 4 and trivial Jones polynomial. It is known that the Jones polynomial is not a complete knot invariant, but it is unknown whether it detects the unknot. So deeper understanding the Burau representation can significantly impact the field of knot theory.

The Jones representations are parameterized by a variable q, though for the Burau representation the variable is typically denoted by a t. The main results in this thesis are about choosing careful specializations of the parameter so that the image is a **discrete** group.

1.5 Motivation For Discreteness

There are two major motivations for discrete representations: the search for thin groups, and Wielenberg's Theorem.

A **lattice** is a discrete subgroup of a Lie Group that has finite co-volume. A thin group can be thought of as a generalization of a lattice. That is, a **thin group** is a Zariski dense, infinite index subgroup of a lattice. One possible approach to find thin groups is to first find discrete representations of a group into a lattice with infinite image. The image is a subgroup of the lattice which has potential to be thin.

A second motivation is Wielenberg's theorem stated below. This theorem gives a way to create faithful representations using sequences of discrete representations. This is particularly interesting in light of the open faithfulness question for the Burau representation.

Theorem 1.5.1 (Wielenberg, [32]). *Let $\rho_i : B_n \to G$ be a sequence of discrete representations, where G is a linear Lie Group. Suppose that*

1. *For each non trivial $\gamma \in B_n$, there exists K_γ so that for $k > K_\gamma$, $\rho_k(\gamma) \neq Id_G$,*

2. *ρ_i converges algebraically to $\rho : B_n \to G$,*

then ρ is faithful, except possibly on the center of B_n

Here, converges algebraically means for each $\omega \in B_n$, $\rho(\omega) = lim_{i \to \infty} \rho_i(\omega)$.

Proof. This proof follows that of Kapovich [16]. Let K be the kernel of ρ. Since B_n is torsion free, then K is torsion free.

Since G is a linear Lie Group, the nilpotency class of its subgroups is bounded above by some constant c. Fix any finite collection $g_1, \cdots, g_k \in K$. Suppose the subgroup $\langle g_1, \cdots, g_k \rangle$ is not nilpotent of class c, then there exists some commutator word of length c, $\omega := [x_1, [x_2, [\dots] \dots] \neq 1$, for $x_i \in \langle g_1, \cdots, g_k \rangle$.

Choosing sufficiently large i, $\rho_i(\omega) \neq Id_G$, and for each g_j, $\rho_i(g_j) \neq Id_G$ and $\rho_i(g_j)$ belongs to the Zassenhaus neighborhood of the identity in G. (A Zassenhaus neighborhood is an open neighborhood Ω of the identity so that every discrete subgroup Δ in G which is generated by $\Delta \cap \Omega$ is contained in a connected nilpotent Lie-subgroup of G.) Since ρ_i is discrete, then $\langle \rho_i(g_1), \cdots \rho_i(g_k) \rangle$ is discrete and generated by elements in the Zassenhaus neighborhood, so the group $\langle \rho_i(g_1), \cdots, \rho_i(g_k) \rangle$ is nilpotent of class c. Since $\rho_i(\omega)$ is a commutator word of length c in $\langle \rho_i(g_1), \cdots, \rho_i(g_k) \rangle$, then $\rho_i(\omega) = Id_G$ contradicting the choice of i.

Similarly, suppose K is not nilpotent of class c. Then there exists some braids $g_1, \cdot, g_k \in K$ and some commutator word ω' of length c so that $\omega'(g_1, \cdots, g_k) \neq 1$. However, the group $\langle g_1, \cdots, g_k \rangle$ is nilpotent of length c, so it must be that $\omega'(g_1, \cdots, g_k) = 1$. Therefore K is nilpotent.

Thus, K is a normal, nilpotent subgroup of B_n, so K must be trivial or central.

□

Chapter 2

The Burau Representation

The Burau representation was first discovered by Werner Burau in 1935 [8]. This representation has garnered much attention over the years for its question of faithfulness. It is well known that the reduced Burau representation is faithful for $n \leq 3$ and unfaithful for $n \geq 5$, but unknown for $n = 4$ [2, 20, 22].

Notation: There are two versions of the Burau representation: reduced and unreduced. The reduced Burau representation is irreducible, while the unreduced is not. For the remainder of this paper, the Burau representation is assumed to be reduced unless otherwise specified.

In addition to its faithfulness intrigue, the Burau representation can also be used to compute the Alexander polynomial of a knot [9, 14]. If a knot K is the closure of a braid ω in B_n, and ρ the Burau representation of B_n, then the Alexander polynomial $\Delta_K(t)$ is

$$\Delta_K(t) = \frac{1-t}{1-t^n} \det(Id - \rho(\omega)).$$

2.1 Definition and Properties

Definition 2.1.1. *The (reduced) Burau representation* $\rho_n : B_n \to GL_{n-1}(\mathbb{Z}[t^{\pm 1}])$ *is given by*

$$\sigma_1 \mapsto \left(\begin{array}{cc|c} -t & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & Id_{n-3} \end{array}\right), \sigma_{n-1} \mapsto \left(\begin{array}{c|cc} I_{n-3} & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & t & -t \end{array}\right)$$

$$\sigma_i \mapsto \left(\begin{array}{c|ccc|c} Id_{i-2} & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & t & -t & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & Id_{n-i-2} \end{array}\right) \text{ for } 2 \leq 1 \leq n-2$$

Squier showed in [28] that there exists a nonsingular $n-1 \times n-1$ matrix J over $\mathbb{Z}[t^{\pm 1}]$ so that for every w in B_n,

$$\rho_n(w)^* J \rho_n(w) = J.$$

Notation: For $M \in GL_{n-1}(\mathbb{Z}[t^{\pm 1}])$, the entries of M are integral polynomials in t and $\frac{1}{t}$, and we denote $M = M(t)$ and $M(\frac{1}{t})$ to be the matrix that replaces t by $\frac{1}{t}$ in the entries of $M(t)$. The involution $*$ is given by $M(t)^* = M(\frac{1}{t})^T$.

Definition 2.1.2. *A **specialization of the Burau representation** is a composition representation $\tau \circ \rho_n$, where $\tau : GL_{n+1}(\mathbb{Z}[t^{\pm 1}]) \to GL_{n+1}(\mathbb{R})$ is an evaluation map determined by $t \mapsto r$ for some fixed $r \in \mathbb{R}$. Typically ρ_n is written at $\rho_{n,t}$ viewing t as a parameter, and the specialization is denoted $\rho_{n,r}$.*

Theorem 2.1.3. *For $r \in \mathbb{C}$, the image of the specialization of the Burau representation at r is isomorphic to the image when specializing to $\frac{1}{r}$. In particular, specializing to r is discrete (faithful) if and only if specializing to $\frac{1}{r}$ is discrete (faithful).*

Proof. Let ψ be the contragradient representation of ρ_n. For $w \in B_n$, if $\rho_n(w) = M(t)$ then $\psi(w) = (M(t)^{-1})^T$ where $M \in GL_{n-1}(\mathbb{Z}[t^{\pm 1}])$. From Squier, there exists a matrix J so that

$$M(t)^* = JM(t)^{-1}J^{-1}.$$

Taking the transpose of both sides shows that $M(\frac{1}{t})$ is conjugate to $(M(t)^{-1})^T$ by J^T.

Thus ρ_n and ψ are conjugate representations. Discreteness and faithfulness is preserved by conjugation, inversion and transposition.

□

2.1.1 Details on Squier's Form

As shown in Theorem 2.1.3, Squier's form is a useful tool for proving structural results about the image of the Burau representation. This section will give detailed computation proofs for the following two results, which are necessary for later use. Letting J denote Squier's form,

1. In dimension $n \geq 4$, $\det J = (1+t)^n \frac{(t^{n+1}-1)}{(t^n(t-1))}$.

2. In dimension n, J may be chosen so it is positive definite for complex values of $t = e^{i\theta}$ for $|\theta| < \frac{2\pi}{n+1}$.

For notational clarity and in this section only, since the following arguments rely heavily on the parameter t, J_t will be used to denote J.

Remark 2.1.4. *J_t is hermitian when t is real or on the unit circle.*

The following computations provide a change of basis to diagonalize J_t into a format useful for analyzing its signature.

Let e_i's be the standard basis vectors in $\mathbb{C}^n$ and $K_i = \frac{1+t+\cdots+t^{i-2}}{1+t+\cdots+t^{i-1}}$. Define a new basis for $\mathbb{C}^n$ as follows

$$\begin{aligned} v_1 &= \{\frac{1}{1+t}, 1, 0, \cdots, 0\} \\ v_2 &= e_1 \\ v_3 &= e_3 + K_3 v_1 \\ v_i &= e_i + K_i v_{i-1} \quad \text{for } 3 < i \leq n \end{aligned}$$

Proposition 2.1.5. *Let S be the matrix whose columns are the v_j's. Then $-S^*J_tS$ is a diagonal matrix with k'th entry equal to $\frac{(1+t)(t^{k+1}-1)}{t(t^k-1)}$ for $k \geq 3$ and first two entries equal to $-\frac{1+t+t^2}{t}$ and $-\frac{(1+t)^2}{t}$.*

This proposition follows from the following computational claims.

Definition 2.1.6. *$\langle x, y\rangle_{J_t} = x^*J_ty$ is an antilinear form on $\mathbb{C}$.*

Claim 2.1.7. *$\langle v_i, v_j\rangle_{J_t} = 0$ for $i \neq j$.*

Proof. $\langle v_1, v_2\rangle_{J_t} = v_1^*J_tv_2 = \frac{1}{1+\frac{1}{t}}b + a = \frac{t}{1+t}(-2-t-\frac{1}{t}) + 1 + t = 0$

Since $v_i = e_i + K_iv_{i-1}$, it suffices to prove that $\langle v_i, v_{i-1}\rangle_{J_t} = 0$ for $1 \geq 3$.

$\langle v_2, v_3\rangle_{J_t} = v_2^*J_tv_3 = e_1J_t(e_3 + K_3v_1) = e_1J_te_3 + K_3e_1J_tv_1 = \{b, c, 0\}e_3 + K_3(b\frac{1}{t+1} + c)$
$= K_3(-\frac{(t-1)^2}{t}\frac{1}{t+1} + \frac{t+1}{t})$

□

Claim 2.1.8. *$\langle v_1, v_1\rangle_{J_t} = -\frac{1+t+t^2}{t}$ and $\langle v_2, v_2\rangle_{J_t} = -\frac{(1+t)^2}{t}$.*

Proof. $\langle v_1, v_1\rangle_{J_t} = v_1^*J_tv_1 = [b\frac{t}{1+t} + a, c\frac{t}{1+t} + b, c, 0, \cdots, 0]v_1 = b\frac{t}{1+t}\frac{1}{1+t} + a\frac{1}{1+t} + c\frac{t}{1+t} + b$

$$= -\frac{(t+1)^2}{t}\frac{t}{(1+t)^2} + 1 + 1 - \frac{(t+1)^2}{t} = \frac{t-(1+t)^2}{t} = -\frac{1+t+t^2}{t}.$$

$$\langle v_2, v_2\rangle_{J_t} = v_2^*J_tv_2 = [b, c, 0, \cdots, 0]v_2 = b = -\frac{(1+t)^2}{t}.$$ □

Claim 2.1.9. *$\langle v_k, v_k\rangle_{J_t} = \frac{-(1+t)(t^{k+1}-1)}{t(t^k-1)} = \frac{-(1+t)}{t}K_{k+1}^{-1}$, for $k \geq 3$.*

Proof.

$$\begin{aligned}\langle v_k, v_k\rangle_{J_t} &= v_k^*J_tv_k = (K_k^*v_{k-1} + e_k^*)J_t(K_kv_{k-1} + e_k)\\ &= K_k^*v_{k-1}^*J_tK_kV_{k-1} + K_k^*v_{k-1}^*J_te_k + e_k^*J_tK_kv_{k-1} + e_k^*J_te_k\\ &= K_kK_k^*(v_{k-1}^*J_tv_{k-1}) + K_k^*v_{k-1}^*(0, \cdots, 0, c, b) + K_k(0, \cdots, 0, a, b)v_{k-1} + b\\ &= K_kK_k^*(v_{k-1}^*Jv_{k-1}) + K_k^*c + K_ka + b \qquad (*)\end{aligned}$$

Now $K_k = \frac{1+t+\cdots+t^{k-2}}{1+t+\cdots+t^{k-1}} = \frac{t^{k-1}-1}{t^k-1}$ and so $K_k^* = \frac{t(t^{k-1}-1)}{t^k-1} = tK_k$. Also, by inductive hypothesis

$$v_{k-1}^*J_tv_{k-1} = \frac{-(1+t)}{t}K_k^{-1}.$$

Thus from $(*)$ we get

$$\begin{aligned}
\langle v_k, v_k\rangle_{J_t} &= K_k K_k^*(-t^{-1}(1-t)K_k^{-1}) + tK_k c + k_n a + b \\
&= -(1+t)K_k + tK_k \frac{1+t}{t} + K_k a + b \\
&= K_k a + b \\
&= \frac{t^{k-1}-1}{t^k-1}(1+t) - \frac{(1+t)^2}{t} \\
&= \frac{(1+t)(t(t^{k-1}-1) - (1+t)(t^k-1))}{t(t^{k+1}-1)} \\
&= -\frac{(1+t)(t^{k+1}-1)}{t(t^k-1)} = \frac{-(1+t)}{t} K_{k+1}^{-1}
\end{aligned}$$

$\square$

These claims prove Proposition 2.1.5. For notational clarity, temporarily let $\mathbb{J}_{n,t} = -S^* J_t S$, where the n denotes the dimension of the matrices.

Corollary 2.1.10.

$$\begin{aligned}
\det \mathbb{J}_1 &= \frac{1+t+t^2}{t}. \\
\det \mathbb{J}_2 &= \frac{1+t+t^2}{t}\frac{(1+t)^2}{t} = \frac{(1+t)^2(1+t+t^2)}{t^2}. \\
\det \mathbb{J}_3 &= \frac{(1+t)^2(1+t+t^2)}{t^2}\frac{(1+t)^2(1+t^2)}{t(1+t+t^2)} = \frac{(1+t)^4(1+t^3)}{t^3}.
\end{aligned}$$

Corollary 2.1.11. $\det \mathbb{J}_{n,t} = (1+t)^n \frac{(t^{n+1}-1)}{(t^n(t-1))}$ *for* $n \geq 4$.

Proof. Induct on n. Base case n=4:

$$\mathbb{J}_4 = \begin{pmatrix} \frac{1+t+t^2}{t} & & & 0 \\ & \frac{(1+t)^2}{t} & & \\ & & \frac{(1+t)^2(1+t^2)}{t(1+t+t^2)} & \\ 0 & & & \frac{1+t+t^2+t^3+t^4}{t(1+t^2)} \end{pmatrix}$$

$\det \mathbb{J}_4 = \frac{1+t+t^2}{t}\frac{(1+t)^2}{t}\frac{(1+t)^2(1+t^2)}{t(1+t+t^2)}\frac{1+t+t^2+t^3+t^4}{t(1+t^2)} = \frac{(1+t)^4(1+t+t^2+t^3+t^4)}{t^4} = \frac{(1+t)^4(t^5-1)}{t^4(t-1)}$

Induction step:

$$\begin{aligned}\det \mathbb{J}_n &= \det \mathbb{J}_{n-1} \frac{(1+t)(t^{n+1}-1)}{t(t^n-1)} \\ &= (1+t)^{n-1} \frac{(t^{n-1+1}-1)}{(t^{n-1}(t-1))} \frac{(1+t)(t^{n+1}-1)}{t(t^n-1)} \\ &= (1+t)^n \frac{t^{n+1}-1}{t^n(t-1)}\end{aligned}$$

□

2.1.2 Signature analysis of Squier's Form

Proposition 2.1.12. *$\mathbb{J}_{n,t}$ is positive definite if and only if $t = e^{i\theta}$ for $|\theta| < \frac{2\pi}{n+1}$.*

Proof. For $t = 1$, it is easily seen that $\mathbb{J}_{n,1}$ is positive definite. Since det is a continuous map, $\mathbb{J}_{n,t}$ can only change signature at the zeros of $\det \mathbb{J}_{n,t}$. The zeros of $\det \mathbb{J}_{n,t}$ are the $n+1$ roots of unity and -1. Thus J_t is positive definite for $t \in R_{>0}$ and $t = e^{i\theta}$ for $|\theta| < \frac{2\pi}{n+1}$.

Let $t = e^{i\theta}$. Consider the eigenvalues of $\mathbb{J}_{3,t}$:

- $\frac{1+t+t^2}{t} > 0$ when $|\theta| < \frac{2\pi}{3}$, and negative elsewhere.

- $\frac{(1+t)^2}{t} > 0$ on all S^1.

- $\frac{(1+t)^2(1+t^2)}{t(1+t+t^2)} > 0$ when $|\theta| < \frac{\pi}{2}$ and $\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$, and negative otherwise.

Thus, $\mathbb{J}_{3,t}$ can only be positive definite for $|\theta| < \frac{\pi}{2} = \frac{2\pi}{3+1}$.

Inductively, assume $J_{n-1,t}$ is only positive definite for $|\theta| < \frac{2\pi}{(n-1)+1}$. $\mathbb{J}_{n,t}$ can be written

$$\mathbb{J}_{n,t} = \left(\begin{array}{c|c} \mathbb{J}_{n-1,t} & 0 \\ \hline 0 & \frac{(1+t)(t^{n+1}-1)}{t(t^n-1)} \end{array}\right).$$

Within the constraint that $|\theta| < \frac{2\pi}{(n-1)+1}$, the last eigenvalue $\frac{(1+t)(t^{n+1}-1)}{t(t^n-1)}$ is only positive when $|\theta| < \frac{2\pi}{n+1}$. Thus $\mathbb{J}_{n,t}$ is only positive definite for $|\theta| < \frac{2\pi}{n+1}$.

Moreover, since each eigenvalue has at most one repeated root (at -1), each eigenvalue alternates sign around the circle changing at roots of unity. Thus the signature of $\mathbb{J}_{n,t}$ starts at $(n, 0)$ at $t = 1$ and changes incrementally to $(0, n)$ at(near) $t = -1$.

□

Remark 2.1.13. *If α is a positive real number, then $\alpha\mathbb{J}_{n,t}$ is also positive definite if and only if $t \in R_{>0}$ or $t = e^{i\theta}$ for $|\theta| < \frac{2\pi}{n+1}$.*

2.2 Details on the Burau Representation of B_3

The goal of this Section is to prove a complete classification of the real discrete specializations of the Burau representation of B_3 described in the following theorem.

Theorem 2.2.1. *The real discrete specializations of the Burau representation of B_3 are exactly when t satisfies one of the following:*

1. $t < 0$ *and* $t \neq -1$
2. $0 < t \leq \frac{3-\sqrt{5}}{2}$ *or* $t \geq \frac{3+\sqrt{5}}{2}$
3. $\frac{3-\sqrt{5}}{2} < t < \frac{3+\sqrt{5}}{2}$ *and the image forms a triangle group.*

Additionally, the specialization is faithful in (1) *and* (2).

2.2.1 Subgroup Properties of B_3

There are two well known subgroups of B_3 that play a vital role in the classification.

1. The center of B_3 is $Z(B_3) = \langle(\sigma_1\sigma_2)^3\rangle$ which is cyclic.

2. The normal subgroup $N = \langle a_1, a_2 \rangle$ where $a_1 = \sigma_1^{-1}\sigma_2$ and $a_2 = \sigma_2\sigma_1^{-1}$, which is a free group on two generators. A proof of this will be shown in the proof of Theorem 2.2.1.

These subgroups will be used in combination with the following Lemmas and Theorem.

Lemma 2.2.2 (Long [21]). *Let $\rho : B_n \to GL(V)$ be a representation and $K \triangleleft B_n$ with K nontrivial and non central. If $\rho|_K$ is faithful, then ρ is faithful except possibly on the center.*

Lemma 2.2.3. *Every homomorphism ϕ on N with $\phi(N)$ a free group of rank two is an isomorphism onto its image.*

Proof. Since N is a free group of rank two, it is Hopfian. It is given that $\phi(N)$ is also a free group of rank two. Therefore by definition of Hopfian, ϕ must be an isomorphism on N.

□

Definition 2.2.4. *The Burau representation of B_3 is the homomorphism $\rho_3 : B_3 \to GL_2(\mathbb{Z}[t, t^{-1}])$ given by*

$$\rho_3(\sigma_1) = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad \rho_3(\sigma_2) = \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}.$$

Lemma 2.2.5. *The Burau representation of B_3 is faithful on the center for all real specializations of t except $t = 0, \pm 1$.*

Proof. The center of B_3 is cyclicly generated by $(\sigma_1\sigma_2)^3$, where

$$\rho_3\left((\sigma_1\sigma_2)^3\right) = \begin{pmatrix} t^3 & 0 \\ 0 & t^3 \end{pmatrix}.$$

This shows that $\rho_3(Z(B_3))$ is a free group on one generator when $t \neq \pm 1, 0$. So ρ_3 is faithful on $Z(B_3)$. □

Corollary 2.2.6. *Away from 0 and* ± 1*, if a specialization the Burau representation is faithful on* N*, then it is faithful on all of* B_3*.*

Proof. Lemma 2.7 proves that the specialization is faithful on the center. Since N is a normal subgroup of B_3, Lemma 2.5 guarantees that the specialization is faithful on the rest of B_3. □

Theorem 2.2.7. *If* ρ_3 *is discrete on* N*, then* ρ_3 *is discrete on all of* B_3*.*

Proof. Assume for a contradiction that $\{\gamma_k\}$ is a sequence in $\rho_3(B_3)$ converging to the identity but $\gamma_k \neq Id$ for all k. Then for every fixed $\phi \in \rho_3(N)$, the commutator sequence $\{[\phi, \gamma_k]\}$ also converges to the identity. Since N is normal and $\rho_3(N)$ is discrete, then $\{[\phi, \gamma_k]\} \subseteq \rho_3(N)$ and for some $n_0 \in \mathbb{N}$, $[\phi, \gamma_k] = Id$ for all $k > k_0$. This gives that for all $k > n_0$,

$$\phi\gamma_k = \gamma_k\phi.$$

This shows that every $\phi \in \rho_3(N)$ commutes with γ_k for large k, and further ϕ and γ_k have the same fixed points. Because B_3 is not virtually solvable, $\rho_3(B_3)$ is non-elementary and $\rho_3|_N$ is discrete, there exists two hyperbolic element η and ϕ of $\rho_3(N)$ so that ϕ and η have different fixed points [26, p. 606]. This contradicts the fact that both ϕ and η must have the same fixed points as γ_k for large enough k.

□

Remark: Theorem 2.2.7 can be generalized with effectively the same proof, but is a slight tangent from the realm of braids and requires a bit of hyperbolic geometry.

Theorem 2.2.7 generalized: *Let* G *be a group that is not virtually solvable and* K *a non central normal subgroup of* G*. If* $\rho : G \to Isom^+(\mathbb{H}^n)$ *is a homomorphism so that* $\rho(G)$ *is non-elementary,* $\rho|_K$ *is discrete, and* $\rho(K) \not\subset Ker(\rho)$ *then* ρ *is discrete on all of* G*.*

2.3 Complete Classification of the Real Discrete Specializations of the Burau Representation of B_3

Theorem 2.2.1 *The real discrete specializations of the Burau representation of B_3 are exactly when t satisfies one of the following:*

1. $t < 0$ *and* $t \neq -1$

2. $0 < t \leq \frac{3-\sqrt{5}}{2}$ *or* $t \geq \frac{3+\sqrt{5}}{2}$

3. $\frac{3-\sqrt{5}}{2} < t < \frac{3+\sqrt{5}}{2}$ *and the image forms a triangle group.*

Additionally, the specialization is faithful in (1) *and* (2)*.*

Proof. With the aim to apply Theorem 2.2.7, the image of the normal subgroup N under ρ_3 is generated by the following two matrices.

$$\rho_3(a_1) = \begin{pmatrix} \frac{t-1}{t} & -1 \\ t & -t \end{pmatrix} \qquad \rho_3(a_2) = \begin{pmatrix} -\frac{1}{t} & \frac{1}{t} \\ -1 & 1-t \end{pmatrix}$$

Next, define ι, x and y as follows

$$\iota = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$x = \iota^{-1}\rho_3(a_2)\iota = \begin{pmatrix} -\frac{1}{t} & -\frac{1}{t} \\ 1 & 1-t \end{pmatrix}, \quad \text{and} \quad y = \iota^{-1}\rho_3(a_1)\iota = \begin{pmatrix} \frac{t-1}{t} & 1 \\ -t & -t \end{pmatrix}.$$

Let S_t denote the specialization of ρ_3 for some fixed $t \in \mathbb{R}$ and $M = \langle x, y \rangle$ in $GL_2(\mathbb{R})$. Since $S_t(N)$ is conjugate to M by ι, the discreteness of $S_t(N)$ is completely determined by the discreteness of M.

Let $D^2 = \mathbb{H}^2 \cup S^1_\infty$ denote the Poincare disk model of the upper half plane. Notice that $x, y \in SL_2(\mathbb{Z}[t, \frac{1}{t}])$ and $\text{tr}(x) = \text{tr}(y) = -\frac{1}{t} + 1 - t$. By comparing $(-\frac{1}{t} + 1 - t)^2$ to 4, both x and y act as isometries of the following type:

1. Hyperbolic when $t < 0$ or $0 < t < \frac{3-\sqrt{5}}{2}$ or $t > \frac{3+\sqrt{5}}{2}$,

2. Elliptic when $\frac{3-\sqrt{5}}{2} < t < \frac{3+\sqrt{5}}{2}$,

3. Parabolic when $t = \frac{3\pm\sqrt{5}}{2}$.

Consider the following cases on $t \in \mathbb{R}$.

Case 1) Let $t < 0$.

In this range of t, both x and y act as hyperbolic isometries on D^2. Consider the following images of ∞:

$$y^{-1}(\infty) = -1, \quad \text{and} \quad xy^{-1}(\infty) = 0$$

$$yxy^{-1}(\infty) = -\frac{1}{t} = x(\infty).$$

The shaded region of Figure 2.1 is a fundamental domain for the action of M on D^2. So $\mathbb{H}^2/M$ is a punctured torus, showing that M and $S_t(N)$ are discrete, and M is a free group of rank 2. By Theorem 2.2.7, since S_t is discrete on N then it is discrete on all of B_3.

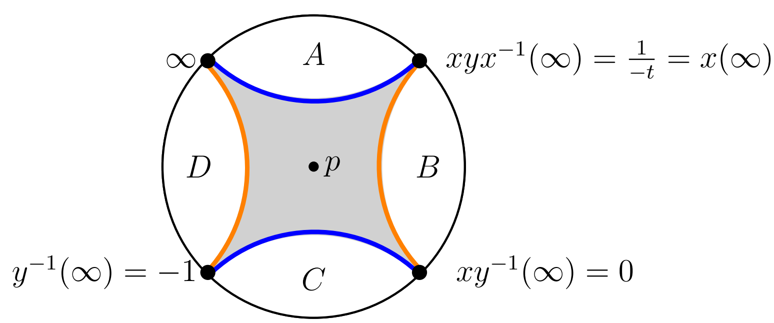


Figure 2.1: D^2 with geodesics connecting images of ∞, when $t < 0$.

To see why the action is discrete, it suffices to show that the center point p can never be fixed by an element of M. Let A, B, C and D be the un-shaded regions in the disk bounded by the geodesics as shown in Figure 2.1. Notice that $x(p) \in B$, $x^{-1}(p) \in D$, $y(p) \in A$ and $y^{-1}(p) \in C$. Similarly, for any integer n, $x^n(p) \in D \cup B$ and $y^n(p) \in A \cup C$. Lastly, $x^n(A \cup C) \subset D \cup B$ and $y^n(D \cup B) \subset A \cup C$. Any element in M is of the form

$x^{e_1}y^{e_2}\cdots x^{e_{m-1}}y^{e_m}$ for some $e_i \in \mathbb{Z}$, giving that $x^{e_1}y^{e_2}\cdots x^{e_{m-1}}y^{e_m}(p) \in A\cup B\cup C\cup D$ and could not possibly fix p.

Case 2) Let $t = \frac{3+\sqrt{5}}{2}$.

For this value of t, x, y and yx^{-1} are parabolic isometrics. Let x_f^{-1}, y_f and z_f denote fixed points of x^{-1}, y and yx^{-1} respectively. By computing eigenvectors, these fixed points are

$$x_f^{-1} = \frac{-1+\sqrt{5}}{2}, \quad y_f = \frac{1-\sqrt{5}}{2}, \quad z_f = \frac{-7+3\sqrt{5}}{2}.$$

Figure 2.2 shows a fundamental domain for the action of M on D^2, showing that $\mathbb{H}^2/M$ is a thrice punctured sphere. By the same arguments as in case 1, S_t is discrete and faithful on all of B_3.

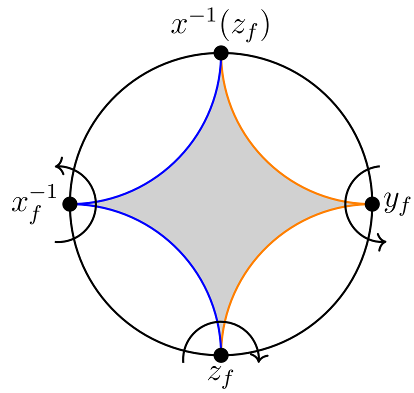


Figure 2.2: The shaded region is the fundamental domain for the action of M on D^2 when $t = \frac{3+\sqrt{5}}{2}$.

Case 3) Let $t > \frac{3+\sqrt{5}}{2}$.

In this region, both x, y, yx and yx^{-1} act as hyperbolic isometries on the D^2. As shown in Case 2, the fixed points of x^{-1}, y and yx^{-1} are distinct when $t = \frac{3+\sqrt{5}}{2}$. If there exists a t so that any two of x^{-1}, y or $x^{-1}y$ shared a fixed point then, then both x^{-1} and y share a fixed point. In other words, x^{-1} and y have a common eigenvector and are simultaneously conjugate to matrices of the form

$$x^{-1} \sim \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \quad \text{and} \quad y \sim \begin{pmatrix} b & * \\ 0 & b^{-1} \end{pmatrix}$$

for some $a, b \in \mathbb{R}$. This forces the commutator $[x^{-1}, y]$ to have the form

$$[x^{-1}, y] \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix},$$

which gives $\mathrm{tr}([x^{-1}, y]) = 2$. However, by direct computation, $\mathrm{tr}([x^{-1}, y]) = \frac{(1+t^2)(1-t^2+t^4)}{t^3}$ which is strictly greater than 2 for $t > \frac{3+\sqrt{5}}{2}$. So as t increases, all six fixed points of x^{-1}, y and $x^{-1}y$ remain distinct for all $t > \frac{3+\sqrt{5}}{2}$.

Let $x_\pm$, and $y_\pm$ denote the fixed points of each x, y respectively. Since x, y, yx, and yx^{-1} are all hyperbolic in this interval for t, there exists disjoint geodesics about each of $x_\pm$ and $y_\pm$ as shown in Figure 2.3. The action of M on D^2 shows that $\mathbb{H}^2/M$ is a pair of pants, and thus S_t is discrete and faithful on B_3.

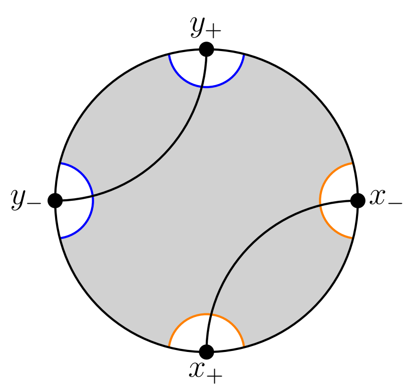


Figure 2.3: The shaded region is the fundamental domain for the action of M on D^2 when $t > \frac{3+\sqrt{5}}{2}$.

Case 4) Let $0 < t \leq \frac{3-\sqrt{5}}{2}$.

Immediately from case 2, case 3, and Theorem 2.2.7, S_t is discrete and faithful on all of B_3.

Case 5) Let $\frac{3-\sqrt{5}}{2} < t < \frac{3+\sqrt{5}}{2}$.

In this region, x^{-1}, y and yx^{-1} are all elliptic with the same trace $1 - t - \frac{1}{t}$. Elliptic isometries are diagolizable with diagonal entries complex conjugate roots of unity. So the trace is $2\cos\theta$ for some θ which is the rotation angle for the isometry. At $t = \frac{3+\sqrt{5}}{2}$, the trace of x^{-1}, y and yx^{-1} are all equal to -2. To account for this negative sign, the following equation must hold

$$-2\cos\theta = 1 - t - \frac{1}{t}.$$

Solving for t in terms of θ gives

$$t = \frac{1 + 2\cos\theta \pm \sqrt{(2\cos\theta + 1)^2 - 4}}{2}.$$

Since t is real valued, the discriminant must be nonnegative, forcing

$$\cos\theta \leq -\frac{3}{2} \qquad \text{or} \qquad \cos\theta \geq \frac{1}{2}.$$

Thus, the only possible rotation angles for x^{-1}, y and yx^{-1} are $0 \leq \theta \leq \frac{\pi}{3}$ or $\frac{5\pi}{3} \leq \theta \leq 2\pi$. Consider the following cases for θ.

1. If $\theta = d\pi$ where d is irrational.

   Let x_f and y_f be the fixed points of x and y respectively. Since y acts as a rotation about y_f, the set $\{y^i(x_f)\}_{i\in\mathbb{N}}$ lies in an S^1 centered at y_f. Since $\frac{\theta}{\pi}$ is irrational, $y^i(x_f)$ is distinct for each i. By compactness, $\{y^i(x_f)\}_{i\in\mathbb{N}}$ has an accumulation point, giving the orbit of x_f is not discrete and the action of M is not discrete.

2. If $\theta = \frac{2\pi}{n}$ for some $n \in \mathbb{Z}$.

   Then M is the triangle group with presentation $\langle x, y | x^n = y^n = (xy)^n = 1\rangle$. The bounds for θ force $n \geq 6$ and all such n occur from specializations of t satisfying $\frac{3-\sqrt{5}}{2} < t < \frac{3+\sqrt{5}}{2}$. For $n \geq 6$, $\frac{1}{n} + \frac{1}{n} + \frac{1}{n} < 1$ so M is a hyperbolic triangle group and is known to be discrete.

3. If $\theta = \frac{2\pi k}{m}$ for $k, m \in \mathbb{Z}$ relatively prime.

   The classification of good orbifolds gives that D^2/M can not yield a cone angle of $\frac{2\pi k}{m}$ for $k, m \in \mathbb{Z}$ relatively prime. So the action of M is not discrete.

□

2.4 Corollaries and Examples

There is interesting faithfulness interplay between the Burau representations ρ_3 on B_3 and ρ_4 on B_4. The underlying reason for this interplay is the block structure of ρ_4 shown in the definition below.

$$\rho_4(\sigma_1) = \begin{pmatrix} -t & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\rho_3(\sigma_1)$}} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right),$$

$$\rho_4(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{pmatrix} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$\rho_3(\sigma_2)$}} & 0 \\ & & 1 \\ \hline 0 & 0 & 1 \end{array}\right),$$

$$\rho_4(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & -t \end{pmatrix}.$$

One way to create an unfaithful specialization of ρ_4 is to "extend" an unfaithful specialization of ρ_3. More precisely, suppose the specialization of ρ_3 at η is unfaithful, and let K denote the kernel in B_3. We can identify K as a subgroup of B_4 under the standard inclusion. From the block structures shown above, $\rho_4(K)$ consists of upper triangular matrices with ones along the diagonals, which is a nilpotent group as a subgroup of the Heisenberg group. Thus the upper central series finitely terminates yielding a nontrivial subgroup of K that maps to the identity by ρ_4. Therefore, the specialization of ρ_4 at η is also unfaithful.

Example 4.1 shows one method to create unfaithful specializations of ρ_3, which consequently are also unfaithful specializations of ρ_4. Because of this consequential relationship, it is perhaps more interesting to find an unfaithful specialization of ρ_4 that is faithful when restricted to B_3. Example 2.4.1 gives a construction of such a specialization.

Example 2.4.1. *A method to create unfaithful specializations of ρ_3 on B_3.*

Let w be a word in B_3 different from σ_1^k. Let f_w be a polynomial factor of the 2-1 entry of $\rho_3(w)$ and t_w be a root of f_w. Specializing to $t = t_w$ leaves $S_{t_w}(w)$ an upper triangular matrix. Since the image of σ_1 is also upper triangular, the group $\langle S_{t_w}(\sigma_1), S_{t_w}(w)\rangle$ is solvable. Therefore, specializing to t_w cannot be faithful since B_3 does not have solvable subgroups.

Some examples such w's and f_w's are listed here.

1. Let $w = \sigma_2^{-2}\sigma_1\sigma_2^{-1}$ with $f_w = -1 + t - 2t^2 + t^3$ which has one real root.

2. Let $w = \sigma_2^5\sigma_1^2\sigma_2^{-4}\sigma_1\sigma_2^3$ and

   $f_w = 1 - 3t + 6t^2 - 10t^3 + 13t^4 - 16t^5 + 16t^6 - 15t^7 + 12t^8 - 8t^9 + 5t^{10} - 3t^{11} + t^{12}$

   which has two real roots.

Theorem 2.2.1 proved that all real unfaithful specializations of ρ_3 come from the interval $(\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2})$. Thus we can conclude that all real roots of f_w must lie in the interval $(\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2})$. This proves the following corollary.

Corollary 2.4.2. *Real roots of the 2-1 entries of Burau matrices not in $\langle\sigma_1\rangle$ must lie in the interval $(\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2})$.*

Example 2.4.3. *An unfaithful specialization of ρ_4 on B_4.*

For simplification, let $x = \sigma_1\sigma_3^{-1}$ and $y = \sigma_2 x \sigma_2^{-1}$. Consider the following words

$$\omega_1 = x^{-1}y^2x^{-1}yxyx^2y^{-2}x^{-1}y^{-3} \tag{2.1}$$
$$\omega_2 = y^{-1}xy^{-2}xy^{-1}x^{-1}y^{-1}x^{-2}y^2xy^2. \tag{2.2}$$

One can check that $\rho_4(\omega_1) \neq \rho_4(\omega_2)$. However, for S_{t_0} the specialization of ρ_4 to $t_0 = \frac{3+\sqrt{5}}{2}$, the equality $S_{t_0}(\omega_1) = S_{t_0}(\omega_2)$ occurs. Theorem 2.2.1 proved that specializing ρ_3 at t_0 is faithful. Thus, the infidelity of ρ_4 at t_0 is truly a property of B_4, not a consequence of containing B_3.

2.4.1 Moving forward from B_3

Keeping inline with the previous discussions of discreteness, Squier's form easily gives the next result.

Proposition 2.4.4. *The image of the specialization of the Burau representation is discrete at real quadratic algebraic units with positive norm.*

Proof. Let α be a real quadratic algebraic unit with positive norm and σ be the generator of the Galois group of $\mathbb{Q}(\alpha)$. The map σ is determined by $\sigma(\alpha) = \alpha^{-1}$, since α has positive norm. Fix arbitrary n and consider the Burau representation on B_n specialized at α, and J the associated Squier's form. Let $\{A_k\}$ be a sequence of matrices in the image of this specialization and assume that $\{A_k\}$ converges to the Id. Each A_k has entries in $\mathbb{Q}(\alpha)$, so the defining relation of Squier's form $A_k^* J A_k = J$ becomes $(A_k^\sigma)^T = J A_k^{-1} J^{-1}$. So if $A_k \to Id$ then so does A_k^σ. Since σ is *the only* field automorphism, the entires $(A_k)_{ij}$ are all algebraic integers of bounded absolute value and degree. There are only finitely many such algebraic integers, so the entries $(A_k)_{ij}$ must be eventually constant. □

Corollary 2.4.5. *The specialization of the Burau representation of* B_3 *at* $\frac{3+\sqrt{5}}{2}$ *is discrete.*

The number $\frac{3+\sqrt{5}}{2}$ is particularly interesting as ρ_3 specialized at $\frac{3+\sqrt{5}}{2}$ is both discrete and faithful, while specializing ρ_4 at $\frac{3+\sqrt{5}}{2}$ is discrete and yet unfaithful.

The discreteness in Theorem 2.2.1 required specific characteristics of B_3 and the fact that the Burau representation is 2-dimensional. However, Proposition 2.4.4 only required

Squier's form and no limitations of the dimension of the representation. Proposition 2.4.4 is motivation for a larger class of discrete representations using Salem numbers and generalized unitary groups, which will be described in the next chapters.

Chapter 3

Discrete Generalized Unitary Groups

3.1 Generalized Unitary Groups

A matrix is *unitary* over the complex numbers if $\overline{M}^{\intercal}M = Id$, where $\overline{M}$ is the complex conjugate of M. We can rewrite this as $\overline{M}^{\intercal} \cdot Id \cdot M = Id$. The collection of all unitary matrices over $\mathbb{C}$ gives the *unitary group*, denoted

$$U(Id, -, \mathbb{C}) =: \{M \in GL(\mathbb{C}) | \overline{M}^{\intercal} \cdot Id \cdot M = Id\}.$$

We can generalize this group to use an arbitrary coefficient ring R, and an order two automorphism ϕ of R.

$$U(Id, \phi, R) =: \{M \in GL(R) | \phi(M)^{\intercal} \cdot Id \cdot M = Id\}$$

When the automorphism ϕ is understood, we will denote $M^* = \phi(M)^{\intercal}$. To generalize further, if J is a matrix satisfying $J^* = J$, then we can get

$$U(J, \phi, R) =: \{M \in GL(R) | M^*JM = J\}.$$

Definition 3.1.1. *$U(J, \phi, R) =: \{M \in GL(R) | M^*JM = J\}$ is called the generalized unitary group, where $M^* = \phi(M)^\intercal$ and $J^* = J$.*

Here J is called a **sesquilinear form** and if a representation has image in such a generalized unitary group, it is called a **sesquilinear** representation. This generalized unitary group can be thought of as the collection of matrices that preserve an inner product given by

$$\langle v, w \rangle = v^* J w.$$

Example 3.1.2. *Let $R = \mathbb{Q}(\sqrt{5})$ and ϕ be the field automorphism defined by $\sqrt{5} \mapsto -\sqrt{5}$. For J and X below, $X \in U_2(J, \phi, \mathbb{Q}(\sqrt{5}))$.*

$$J = \begin{pmatrix} -10 & 5+\sqrt{5} \\ 5-\sqrt{5} & 10 \end{pmatrix} \qquad X = \begin{pmatrix} \frac{-3+\sqrt{5}}{2} & 1 \\ 0 & 1 \end{pmatrix}$$

Example 3.1.3. *How does this apply to the Burau representations? Squier showed that the Burau representations are sesquilinear with respect to Squier's form J. Letting ϕ be the involution given by $t \mapsto \frac{1}{t}$, we can write*

$$\rho_n : B_{n+1} \to U_n(J, \phi, \mathbb{Z}[t^{\pm 1}]).$$

3.2 Discrete Generalized Unitary Groups

Discreteness of a unitary group is a balance between the form J and the choice of coefficient ring.

Example 3.2.1. *Proposition 2.4.4 can be restated as follows: For α a real quadratic algebraic unit with positive norm, $\mathcal{O}_{\mathbb{Q}(\alpha)}$ the ring of integers for $\mathbb{Q}(\alpha)$, ϕ the map that sends $\alpha \mapsto \frac{1}{\alpha}$, and J a nondegenerate form over $Fix(\phi)$, then the generalized unitary group $U_m(J, \phi, \mathcal{O}_{\mathbb{Q}(\alpha)})$ is discrete.*

This example of discreteness can be extended to a larger class of number rings with greater than quadratic dimension. Let L be a totally real algebraic field extension of

$\mathbb{Q}$ and K be a degree two field extension of L. Let ϕ be the order two generator of $Gal(K/L)$ and $\mathcal{O}_K$, respectively $\mathcal{O}_L$, denote the ring of integers of K and L.

$$\begin{array}{ccc} \mathcal{O}_K \circlearrowleft \phi & K \circlearrowleft \phi & K^\sigma \circlearrowleft \phi^\sigma \subseteq \mathbb{C} \\ \Big| & \Big| \, 2 & \\ \mathcal{O}_L & L & L^\sigma \subseteq \mathbb{R} \end{array}$$

Let σ be a complex place of K, which in this setting is a field homomorphism $\sigma : K \to \mathbb{C}$ different from the identity map. We denote $X^\sigma = \sigma(X)$ for any X in K. The algebraic structure is passed along by σ, meaning $\mathcal{O}_{K^\sigma} = (\mathcal{O}_K)^\sigma$ is the ring of integers for K^σ and $\phi^\sigma = \sigma\phi\sigma^{-1}$ is the involution on K^σ.

Let J be a matrix over $\mathcal{O}_K$ that is sesquilinear with respect to ϕ. Since the fixed field of ϕ is L, J must have diagonal entries in L. J^σ is sesquilinear with respect to ϕ^σ. So in particular,

$$U_m(J^\sigma, \phi^\sigma, \mathcal{O}_{K^\sigma}) = \{M \in GL_m(\mathcal{O}_{K^\sigma}) | (M^{\phi^\sigma})^\intercal J^\sigma M = J^\sigma\}.$$

Since σ is a homomorphism, we can see that $(U_m(J, \phi, \mathcal{O}_K))^\sigma = U_m(J^\sigma, \phi^\sigma, \mathcal{O}_{K^\sigma})$ by applying σ to the equation $J = M^* JM$.

The following results outline compatibility requirements between J and $\mathcal{O}_K$, which result in $U_m(J, \phi, \mathcal{O}_K)$ as a discrete subgroup of $GL_m(\mathbb{R})$, under the standard euclidean topology.

Proposition 3.2.2. *$U_m(J^\sigma, \phi^\sigma, \mathcal{O}_{K^\sigma})$ is a bounded group when J^σ is positive definite, and ϕ^σ is complex conjugation.*

Proof. Because J^σ is positive definite, by Sylvester's Law of Inertia and the Gram-Schmidt process, there exists a matrix $Q \in GL_m(\mathbb{C})$ so that $J^\sigma = Q^* IdQ$. This implies that $QU_m(J^\sigma, \phi^\sigma, \mathcal{O}_{K^\sigma})Q^{-1} \subseteq U_m(Id, \phi^\sigma, \mathbb{C})$ which is a subgroup of the compact group U_m. □

Theorem 3.2.3. *$U_m(J, \phi, \mathcal{O}_K)$ is discrete if for every non-identity place σ of K, J^σ is positive definite and ϕ^σ is complex conjugacy.*

Proof. Assume that $\{M_n\}$ converges to the identity in $U_m(J, \phi, \mathcal{O}_K)$. To show $\{M_n\}$ is eventually constant, we will show that for n large, there are only finitely many possibilities for the entries $(M_n)_{ij}$.

By assumption, for each σ the group $U_m(J^\sigma, \phi^\sigma, O_{K^\sigma})$ is bounded by Proposition 3.2.2. Also, for every M_n, $M_n^\sigma \in U_m(J^\sigma, \phi^\sigma, O_{K^\sigma})$. So there exists a B so that for large n, for all i, j, and for all σ, that $|(M_n^\sigma)_{ij}| < B$.

For every $M \in U_m(J, \phi, \mathcal{O}_K)$, the equation $M^* J M = J$ can be rearranged to $JMJ^{-1} = ((M^\phi)^\intercal)^{-1}$, showing that M and $((M^\phi)^\intercal)^{-1}$ are simultaneously conjugate. Thus $\{M_n^\phi\}$ also converges to the identity. Convergent sequences are bounded, so for large enough n, $|(M_n)_{ij}| < B$ and $|(M_n)_{ij}^\phi| < B$ for every ij-entry.

L is a totally real degree two subfield of K, and ϕ generates $Gal(K/L)$. So K has one non-identity real embedding ϕ, and all other embeddings are complex. Thus we have shown above that for large n there is a uniform bound B for each entry $(M_n)_{ij}$ and each Galois conjugate of $(M_n)_{ij}$. There are only finitely many algebraic integers α so that $deg(\alpha) \leq deg(K/\mathbb{Q})$, and with the property that α and all of the Galois conjugates of α have absolute value bounded above by B. So there are only finitely many possible entries for $(M_n)_{ij}$, which implies the sequence $\{M_n\}$ is eventually constant. □

Corollary 3.2.4. *If $\rho : G \to U_m(J, \phi, \mathcal{O}_K)$ is a representation of a group G so that for every non-identity place σ of K, J^σ is positive definite and ϕ^σ is complex conjugacy, then ρ is a discrete representation.*

With the Burau representation in mind, Theorem 3.2.3 requires an algebraic unit α so that Squier's form J is positive definite at all of the non-identity embeddings of α, in addition to properties of the number ring of α. Recall from Proposition 2.1.12 that J is positive definite in a neighborhood of 1 on the unit circle. This need motivates the use of Salem numbers in the next section.

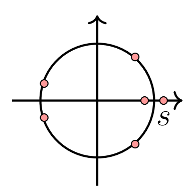
At first glance, the requirements for Corollary 3.2.4 seem very specific and perhaps it is doubtful that any such a representation could exist. However, as described in section

2.1, Squier showed that the Burau representation maps into a generalized unitary group over $\mathbb{Z}[t, t^{-1}]$, so the next task is to find values of t so that so the form and coefficient ring satisfy the specific hypothesis of Corollary 3.2.4. Section 3.4 will show how careful specializations of t to certain Salem numbers meet all of the conditions for Corollary 3.2.4. More generally, Section 4.0.1 will show that every irreducible Jones representation fixes a form J_t with a parameter, and specializations to Salem numbers can also be found to satisfy Corollary 3.2.4.

3.3 Salem Numbers

Salem numbers are the key ingredient to the application of Corollary 3.2.4, which requires a real algebraic number field with tight control and understanding of each of its complex embeddings.

Definition 3.3.1. *A Salem number s is a real algebraic unit greater than 1, with one real Galois conjugate $\frac{1}{s}$, and all complex Galois conjugates have absolute value equal to 1.*

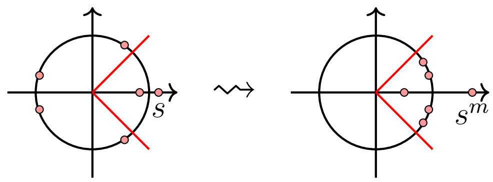


For example, the largest real root of Lehmer's Polynomial, called Lehmer's number,

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1,$$

is a Salem number. Trivial Salem numbers of degree two are solutions to $s^2 - ns + 1$ for $n \in \mathbb{N}$, $n > 2$.It is well known that there are infinitely many Salem numbers of arbitrarily large absolute value and degree. In particular, if s is a Salem number, then s^m is also a Salem number for every positive integer m. One geometric consequence of this property that powers of Salem numbers are Salem numbers, is that by taking powers, one can control the spatial configuration of the complex Galois conjugates of a Salem number, as described in Lemma 3.3.2.

Lemma 3.3.2. *For any interval about 1 on the complex unit circle, there exists infinitely many integers m so that every complex Galois conjugate of s^m lies in the interval.*



Proof. Let $e^{i\theta_1}, \cdots, e^{i\theta_k}$ be all the Galois conjugates of the Salem number s with positive imaginary part. Suppose that $\prod_{j=1}^{k}(e^{i\theta_j})^{m_j} = 1$. Let φ be the automorphism of the Galois closure of s with the property that $\varphi(e^{i\theta_1}) = s$. Since φ must permute the Galois conjugates of s, for $j \neq 1$, $\varphi(e^{i\theta_j})$ is again on the complex unit circle. Thus,

$$1 = \varphi(\prod_{j=1}^{k}(e^{i\theta_j})^{m_j}) = s^{m_1}\prod_{j=2}^{k}\varphi(e^{i\theta_j})^{m_j}, \text{which implies } \prod_{j=2}^{k}\varphi(e^{i\theta_j})^{m_j} = \frac{1}{s^{m_j}}.$$

Since each $\varphi(e^{i\theta_j})$ is a unit complex number, it must be the case that each $m_j = 0$. This shows that the point $p = (e^{i\theta_1}, \cdots, e^{i\theta_k})$ satisfies the criteria for Kronecker's Theorem. In particular, the set $\overline{\{p^m | m \in \mathbb{Z}\}}$ is dense in the torus T^k. $\square$

Fixing an arbitrary Salem number s, let $K = \mathbb{Q}(s)$, $L = \mathbb{Q}(s + \frac{1}{s})$, and $\mathcal{O}_K$ be the the ring of integers of K.

$\mathbb{Q}(s) = K$

| 2

$\mathbb{Q}(s + \frac{1}{s}) = L$

|

$\mathbb{Q}$

Since s and $\frac{1}{s}$ are real and all other Galois conjugates of s are complex, K has exactly two real embeddings. For a complex embedding σ of K, $(s+\frac{1}{s})^\sigma = 2\text{Re}(s^\sigma)$ which is real. This shows that all embeddings of L are real, and that L is a totally real subfield of K. Since s is a root of $X^2 - (s+\frac{1}{s})X + 1$, K is degree two over L.

The Galois group of K/L is generated by ϕ which maps $s \mapsto \frac{1}{s}$. (This exactly matches the involution $t \mapsto \frac{1}{t}$ needed in the sesquilinear condition for the Burau representation.) On the complex unit circle, inversion is the same as complex conjugation. So for the complex embeddings σ of K, ϕ^σ is complex conjugacy. Notice for a sesquilinear matrix J_t over $\mathcal{O}_K$ with a parameter t, specializing $t = s$ leaves J_s^σ hermitian.

Theorem 3.3.3. *Let $\rho_t : G \to GL_m(\mathbb{Z}[t, t^{-1}])$ be a representation of a group G. Suppose there exists a matrix J_t so that:*

1. *$M^* J_t M = J_t$ for all M in the image of ρ_t,*
2. *$J_t = (J_{\frac{1}{t}})^\intercal$,*
3. *J_t is positive definite for t in a neighborhood η of 1.*

Then, there exists infinitely many Salem numbers s, so that the specialization ρ_s at $t = s$ is discrete.

Proof. By Lemma 3.3.2, there are infinitely many Salem numbers with the property that all the complex Galois conjugates lie in η. Let s be one such Salem number. Specializing t to s gives $\rho_s : G \to U_m(J_s, \phi, O_{\mathbb{Q}(s)})$, where ϕ is the usual map given by $s \mapsto \frac{1}{s}$.

Let σ be a complex place of $\mathbb{Q}(s)$ which is given by $s \mapsto z$ for z a complex Galois conjugate of s. Then $J_s^\sigma = J_z$, and since $z \in \eta$, then J_z is positive definite. By Corollary 3.2.4, the specialization ρ_s at $t = s$ is discrete. □

Remark 3.3.4. *If the representations in Theorem 3.3.3 all have determinant 1, then the image is more than just discrete, but in fact is a subgroup of a lattice. See Chapter 6 for more details.*

3.4 Discrete Specializations of the Burau Representation using Salem Numbers

Proposition 3.4.1. *There are infinitely many Salem numbers s so that the Burau representation specialized to $t = s$ is discrete.*

Proof. The specialization of $\rho_{n,1}$ at $t = 1$ collapses to an irreducible representation of the symmetric group. As a representation of a finite group, $\rho_{n,1}$ fixes a positive definite form which is unique up to scaling, by Proposition 4.2.2. At $t = 1$, $J_{n,1}$ is positive definite, and the signature of $J_{n,t}$ can only change at zeroes of its determinant.

By Proposition 2.1.12 and the zeroes of $\det(J_{n,t})$ occur at $n+1$'th roots of unity. Thus, $J_{n,t}$ remains positive definite for unit complex values of t with argument less than $\frac{2\pi}{n+1}$. This shows the reduced Burau representation satisfies the criteria of Theorem 3.3.3. □

Example 3.4.2. *The Burau representation $\rho_{4,t}$ of B_4 is discrete when specializing t to the following Salem numbers:*

- *Lehmer's number raised to the powers 16, 32, and 47,*
- *The largest real root of $1 - x^4 - x^5 - x^6 + x^{10}$ raised to the powers 17, 23, and 43.*

Remark 3.4.3. *Recall Wielenberg's Theorem from Section 1.5. This theorem says that one can create a faithful representation as a limit of discrete representations, with other technical requirements. Since Theorem 3.3.3 gives infinitely many different discrete representations, is it possible that these representations could be used to find a faithful specialization of the Burau representation? More precisely, let $\{s_m\}$ be a sequence of Salem numbers that converge to Salem number s_∞. The specializations of ρ_t at $t = s_m$, $\{\rho_{s_m}\}$, is a sequence of representations that converges to the specialization at $t = s_\infty$, ρ_{s_∞}. If the sequence of s_m's could be chosen so that ρ_{s_m} was discrete, then it is possible that ρ_{s_∞} is a faithful specialization. As you might have guessed, since this is a remark and not a theorem, this convergence is never possible.*

Here's the problem. It is a fact about Salem numbers that if a sequence of Salem numbers converges, their complex Galois conjugates must be dense in the unit circle. However, the discreteness of the specializations in Theorem 3.3.3 requires Salem numbers whose complex Galois conjugates lie in a small region on the unit circle so that the form J is positive definite at those places. So we cannot simultaneously keep discreteness of the specializations and convergence of the Salem numbers.

Chapter 4

The Hecke Algebras and the Jones Representations

The goal of this section is to generalizes Squier's result and show that all of the irreducible Jones representations are sesquilinear, as in the following theorem.

Theorem 4.0.1. *If ρ is an irreducible Jones representation of B_n and q generic unit complex number close to 1, then there exists a non-degenerate, positive definite, sesquilinear matrix J so that for all M in the image of ρ, $(M^{\phi})^{\intercal} J M = J$.*

Then applying Theorem 3.3.3 will give the following discreteness results.

Corollary 4.0.2. *For each irreducible Jones representation, there are infinitely many Salem numbers s so that specializing $q = s$, is a discrete representation.*

Before proving the theorem, there is a brief introduction to the Hecke algebras and Young diagrams establishing only pertinent information from this rich subject.

4.1 Representations of the Hecke Algebras and Young Diagrams

Definition 4.1.1. *The* ***Hecke algebra (of type*** A_n***)****, denoted* $H_n(q)$*, is the complex algebra generated by invertible elements* $g_1, \cdots, g_{n-1}$ *with relations*

$$\begin{aligned} g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} && \text{for all } i < n \\ g_i g_j &= g_j g_i && \text{for } |i-j| > 1 \\ g_i^2 &= (1-q) g_i + q && \text{for all } i < n.(*) \end{aligned} \tag{4.1}$$

Here, q is a complex parameter. $H_n(q)$ is a quotient of $\mathbb{C}[B_n]$ by relation 4.1. This quotient can be seen as an eigenvalue condition which forces the eigenvalues of the generators to be q and -1. In fact, all of the representations of the braid group with two eigenvalues come from representations of the Hecke algebras, see [15]. These representations of the braid group are called the **Jones representations** which are defined by precomposing a representation of $H_n(q)$ by the quotient map from $\mathbb{C}[B_n]$. Notice that there is a standard inclusion of $H_{n-1}(q)$ into $H_n(q)$ by ignoring the last generator. This gives a standard way to restrict a representation of $H_n(q)$ to a representation of $H_{n-1}(q)$, which respects the restriction of B_n to B_{n-1}.

The Hecke algebras come equipped with a natural automorphism, denoted here by ϕ, which sends $q \mapsto \frac{1}{q}$. Taking q to be a unit complex number, this automorphism becomes complex conjugacy. It is easy to see that when $q = 1$, $H_n(q)$ is the complex symmetric group $\mathbb{C}[\Sigma_n]$. What is less obvious but well known is that for q not a root of unity, $H_n(q)$ is isomorphic to $\mathbb{C}[\Sigma_n]$, see [5] pages 54-56. One consequence of this isomorphism is that the parameterization of the irreducible representations of Σ_n by Young diagrams also gives a complete parameterization of the irreducible representations of $H_n(q)$. For a more detailed discussion of Young diagrams see [33], and [31] for a construction of the Jones Representations.

Definition 4.1.2. *A **Young diagram** is a finite collection of boxes arranged in left justified rows, with the row sizes weakly decreasing.*

Every Young diagram contains sub-Young diagrams by removing boxes in a way that retains the weakly decreasing row length condition. If λ is a Young diagram with n boxes, then we will call the sub-Young diagrams found by removing one box from λ the **$(n-1)$-subdiagrams of λ.**

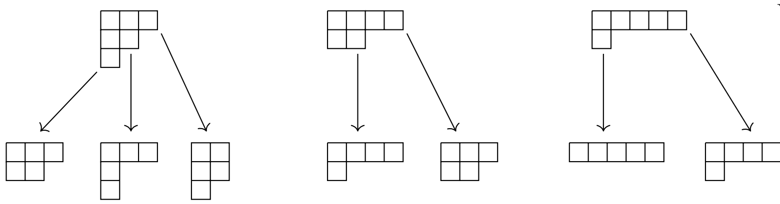


Figure 4.1: Example 5-subdiagrams of three different Young diagrams with 6 boxes.

A Young diagram is completely determined by its list of $(n-1)$-subdiagrams. In fact, a Young diagram is completely determined by any two of its $(n-1)$-subdiagrams. To see this, stack any two $(n-1)$-subdiagrams atop each other top left aligned. Each $(n-1)$-subdiagram will contain the missing box from the other $(n-1)$-subdiagram, recovering the original Young diagram. Notice that each pair of the Young diagrams in Figure 4.1 have one 5-subdiagram in common and it is also possible for two different Young diagrams to have the same number of $(n-1)$-subdiagrams. These $(n-1)$-subdiagrams also determine representations of the Hecke algebras in a powerful way. The following theorem, due to Jones in [15], states concretely the relationship between Young diagrams and the representations of the Hecke algebras.

Theorem 4.1.3. *Up to equivalence, the finite dimensional irreducible representations of $H_n(q)$, for generic q, are in one to one correspondence with the Young diagrams of n boxes. Moreover, if ρ is a representation corresponding to Young diagram λ, then ρ restricted to $H_{n-1}(q)$ is equivalent to the representation $\bigoplus_{i=1}^{k} \rho_{\lambda_i}$ where $\lambda_1, \cdots, \lambda_k$ are all* *of the* $(n-1)$*-subdiagrams of* λ *and each* ρ_{λ_i} *is an irreducible representation of* $H_{n-1}(q)$ *corresponding to* λ_i*.*

Here equivalence means the existence of an intertwining isomorphism made precise by the following definition.

Definition 4.1.4. $\varphi : G \to GL(V)$ *and* $\psi : G \to GL(W)$ *are said to be* ***equivalent*** *representations if there exists a linear isomorphism* $T : V \to W$ *so that* $T\varphi(g)(v) = \psi(g)T(v)$ *for all* $g \in G$ *and* $v \in V$*, or that the following diagram commutes.*

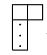
$$\begin{array}{ccc} V & \xrightarrow{\varphi(g)} & V \\ \big\downarrow{\scriptstyle T} & & \big\downarrow{\scriptstyle T} \\ W & \xrightarrow{\psi(g)} & W \end{array}$$

Choosing bases for V and W, the equivalence T gives the matrix equation

$$[T][\varphi(g)][T]^{-1} = [\psi(g)].$$

At the level of matrices, representations are equivalent exactly when they are simultaneously conjugate. In the context of Theorem 4.1.3, the restriction of ρ to $H_{n-1}(q)$ is equivalent to the representation $\bigoplus_{i=1}^{k} \rho_{\lambda_i}$, which means there is a change of basis so that the restriction of ρ is block diagonal.

These restriction rules are combinatorially depicted in the lattice of Young diagrams shown in Figure 4.2. The lines drawn between diagrams in different rows connect the diagrams with n boxes to all of their $(n-1)$-subdiagrams.

Remark 4.1.5. *The lattice of Young diagrams has a chain of diagrams with two columns and only one block in the second column,* *. The representations corresponding to these diagrams are the Burau representations. There is a natural symmetry of the lattice of Young diagrams, so depending on the choice of convention, one could define the Burau representations as the diagrams with exactly two rows, and one box in the second row. The Burau representations are shown in red in Figure 4.2.*

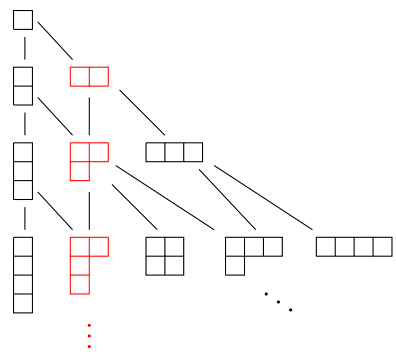


Figure 4.2: Lattice of Young diagrams encoding the restriction rules for the irreducible representations of the Hecke algebras. The Burau representations are shown in red.

4.2 Sesqulinear Representations and Contragredients

As described in Section 3.1, a representation is sesquilinear if there exists an invertible matrix J so that for every M in the image of the representation, the following equation is satisfied

$$M^*JM = J. \tag{4.2}$$

Rearranging this equation, we see that $M = J^{-1}((M^\phi)^\intercal)^{-1}J$ showing that M and $((M^\phi)^\intercal)^{-1}$ are simultaneously conjugate. Changing views slightly, consider the following definition.

Definition 4.2.1. *For $\varphi : G \to GL(V)$ a complex linear representation, $\tilde{\varphi} : G \to GL(V^*)$ is called the **ϕ-twisted contragredient representation** of φ and is given by $\tilde{\varphi}(g)f(v) = f(\varphi(g^{-1})^\phi v)$, for every $g \in G$,$v \in V$ and $f \in V^*$.*

If a basis for V is chosen, then as matrices, $[\tilde{\varphi}(g)] = ([\varphi(g)^\phi]^\intercal)^{-1}$. So another way to view a sesquilinear representation is one that is *equivalent to its ϕ-twisted contragredient.* The reason for using the ϕ-twisting in addition to the contragrediant is to preserve the character of the representation. For example, the Jones representations have eigenvalues -1 and q, and the contragredients representations have eigenvalues -1 and $\frac{1}{q}$. The involution ϕ is necessary to return the $\frac{1}{q}$ eigenvalue back to a q.

This viewpoint combined with the following proposition gives a crucial perspective for the proof of Theorem 4.0.1.

Proposition 4.2.2. *If an absolutely irreducible matrix representation has an invertible matrix J satisfying equation 4.2, then J is unique up to scaling.*

Proof. Suppose there were two such matrices J_1 and J_2. Then equation 4.2 gives for all matrices M in the representation,

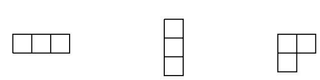
$$\begin{aligned} J_1 M J_1^{-1} = ((M^{\phi})^{\intercal})^{-1} &= J_2 M J_2^{-1} \\ \Rightarrow \quad (J_1^{-1} J_2)^{-1} M (J_1^{-1} J_2) &= M. \end{aligned}$$

This shows that $J_1^{-1} J_2$ is in the centralizer of the entire irreducible representation. Schur's Lemma gives that $J_1^{-1} J_2 = \alpha \cdot \mathrm{Id}$ for some scalar α, and finally $J_2 = \alpha J_1$. □

4.2.1 Proof of Theorem 4.0.1

Lemma 4.2.3. *Every finite dimensional irreducible representation of the Hecke algebra is equivalent to its ϕ-twisted contragediant representation, when q is a generic complex number.*

Proof. We can establish this result for $n = 3$. There are three non-equivalent irreducible representations of $H_3(q)$ corresponding to the following Young diagrams.



Up to equivalence, the first two representations are one dimensional given by $g_i \mapsto q$ and $g_i \mapsto -1$, which are in fact equal to their ϕ-twisted contragredient representations. The third representation is known to be the Burau representation for B_3. As described in Chapter 2, Squier showed that the Burau representations are sesquilinear and are therefore equivalent to their ϕ-twisted contragediant.

Inductively moving forward, let $\rho : H_n(q) \to GL(V)$ be a finite dimensional irreducible representation and $\tilde{\rho}$ be the ϕ-twisted contragredient representation of ρ. Up to equivalence, ρ corresponds to a Young diagram λ. To show that ρ and $\tilde{\rho}$ are equivalent, it suffices to show that both representations correspond to the same λ. A Young diagram is completely characterized by its list of $(n-1)$-subdiagrams, which correspond to the restriction of the representation to $H_{n-1}(q)$. So it is enough to show that the restrictions of ρ and $\tilde{\rho}$ correspond to the same list of $(n-1)$-subdiagrams.

Denoting $\rho| = \rho|_{H_{n-1}(q)}$, by Theorem 4.1.3 there is an equivalence T so that

$$T\rho|(h)T^{-1} = \bigoplus_{i=1}^{k} \rho_{\lambda_i}(h) \qquad \text{for every } h \in H_{n-1}(q),$$

where each λ_i is an $(n-1)$-subdiagram of λ, k is the number of $(n-1)$-subdiagrams of λ, and ρ_{λ_i} is an irreducible representation of $H_{n-1}(q)$ corresponding to λ_i. Choosing a basis for V, the matrix for $[T\rho|(h)T^{-1}]$ is block diagonal. Taking the ϕ-twisted contragredient of a block diagonal matrix preserves the block decomposition, which gives

$$([T^{\phi}]^{\intercal})^{-1}[\tilde{\rho}|(h)][T^{\phi}]^{\intercal} = \bigoplus_{i=1}^{k}[\tilde{\rho}_{\lambda_i}(h)] \text{ for every } h \in H_{n-1}(q).$$

This equation shows that $\tilde{\rho}|$ is equivalent to $\bigoplus \tilde{\rho}_{\lambda_i}$. Since each ρ_{λ_i} is an irreducible representation of $H_{n-1}(q)$, we can inductively assume that ρ_{λ_i} is equivalent to $\tilde{\rho}_{\lambda_i}$, for all $i \leq k$. Therefore, ρ_{λ_i} and $\tilde{\rho}_{\lambda_i}$ correspond to the same Young diagram λ_i. Thus the restrictions of ρ and $\tilde{\rho}$ correspond to the same list of $(n-1)$-subdiagrams.

□

Remark 4.2.4. *While it would be elegant to have a proof of this result using character theory, the generality of this approach allows a broader application to algebras that are not deformations of the symmetric group.*

Theorem 4.0.1. *If ρ is an irreducible Jones representation of B_n and q generic unit complex number close to 1, then there exists a non-degenerate, positive definite, sesquilinear matrix J so that for all M in the image of ρ, $(M^{\phi})^{\intercal}JM = J$.*

Proof. Let ρ be a finite dimensional irreducible representation of $H_n(q)$ over V. By Lemma 4.2.3, ρ is equivalent to its ϕ-twisted contragredient representation $\tilde{\rho}$ by an equivalence T. Choose a basis for V and its dual basis for V^*, let $\mathcal{T}$ be the matrix for T with respect to these bases. We will use this matrix $\mathcal{T}$ to find the desired matrix J. Let superscript $*$ denote the ϕ-twisted transpose of a matrix to ease computation. For all $g \in H_n(q)$, we get the following matrix equations.

$$\begin{aligned} & \mathcal{T}[\rho(g)]\mathcal{T}^{-1} = [\tilde{\rho}(g)] = ([\rho(g)]^{-1})^* \\ \Rightarrow \quad & (\mathcal{T}^{-1})^*[\rho(g)]^*\mathcal{T}^* = [\rho(g)]^{-1} \qquad (\ddagger) \\ \Rightarrow \quad & \mathcal{T}^*[\rho(g)](\mathcal{T}^*)^{-1} = ([\rho(g)]^{-1})^* \end{aligned}$$

This shows that $\mathcal{T}$ and $\mathcal{T}^*$ are two possible forms for ρ. By Proposition 4.2.2, $\mathcal{T} = \alpha\mathcal{T}^*$ for some $\alpha \in \mathbb{C}$. Applying $*$ again gives $\mathcal{T} = \alpha\alpha^*\mathcal{T}$ and $\alpha\alpha^* = 1$.

Define $J = \beta\mathcal{T} + \beta^*\mathcal{T}^* = (\alpha\beta + \beta^*)\mathcal{T}^*$ where β is as follows. (The need for β is to ensure that J is invertible.) If $\alpha \neq -1$, let $\beta = 1$ with gives that $\det J = \det((\alpha+1)\mathcal{T})$ which is nonzero. If $\alpha = -1$, let $\beta \in \mathbb{C}$ so that $\beta^* \neq \beta$. Then $\det J = \det[(\alpha\beta + \beta^*)\mathcal{T}^*] = \det[(-\beta + \beta^*)\mathcal{T}]$ is nonzero. So in both cases, J is invertible.

Secondly, J is sesqulinear, that is $J^* = (\beta\mathcal{T} + \beta^*\mathcal{T}^*)^* = \beta^*\mathcal{T}^* + \beta\mathcal{T} = J$. If M is a matrix in the image of ρ, rearranging equation $(\ddagger)$ gives $M^*\mathcal{T}^*M = \mathcal{T}$. So, inserting J gives

$$M^*JM = M^*(\alpha\beta + \beta^*)\mathcal{T}^*M = (\alpha\beta + \beta^*)M^*\mathcal{T}^*M = (\alpha\beta + \beta^*)\mathcal{T} = J.$$

It remains to show that J is positive definite. Taking $q = 1$, ρ is an irreducible representation of the symmetric group Σ_n. As a linear representation of a finite group, V admits an inner product that is invariant under the action of Σ_n, given by a positive definite nondegenerate matrix $\hat{J}$. Proposition 4.2.2 guarantees that $\hat{J}$ is unique up to scaling. Since $J|_{q=1}$ is also a form for this representation, it must be that $\hat{J}$ is a multiple of $J|_{q=1}$, which gives that J is positive definite for $q = 1$. Since J is Hermitian for unit complex q, it has real eigenvalues, and continuity of the determinant map finally gives that either J or $-J$ is positive definite for q close to 1.

$\square$

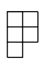
Corollary 4.2.5. *For each irreducible Jones representation, there are infinitely many Salem numbers s so that specializing $q = s^m$, for some m, is a discrete representation.*

4.3 Examples and Computations

Given explicit matrices $S_1, \cdots, S_{n-1}$ for the generators of an irreducible Jones representation of B_n, J can be directly computed by solving the *linear* systems

$$S_i^* J S_i - J = 0$$

for $1 \le i \le n-1$. The form can be made Hermitian by taking $J + J^*$.

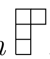
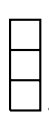
Example 4.3.1. *On page 362 of [15], Jones gives explicit matrices for the irreducible Jones representation of B_6 corresponding to the Young diagram* *, which has only one 5-subdiagram,* *. The restriction to B_5 is also irreducible, and the same form J will work for both the restriction and the full representation. Solving four linear equations as described above yields,*

$$J = \begin{pmatrix} \frac{(1+q)^2}{q} & -1-q & 2 & -1-q & -1-q \\ -\frac{1+q}{q} & \frac{1+q+q^2}{q} & -\frac{1+q}{q} & 1 & 1 \\ 2 & -1-q & \frac{(1+q)^2}{q} & -1-q & -1-q \\ -\frac{1+q}{q} & 1 & -\frac{1+q}{q} & \frac{1+q+q^2}{q} & 1 \\ -\frac{1+q}{q} & 1 & -\frac{1+q}{q} & 1 & \frac{1+q+q^2}{q} \end{pmatrix}.$$

How much can be determined by the decomposition rules? Suppose now that you had the explicit matrices for only the first $n-2$ generators of an irreducible Jones representation of B_n corresponding to Young diagram λ. The decomposition rules determine the matrix for the last generator and the form up to the following variability.

- The matrix for the last generator is unique up to conjugation by any γ in the centralizer of $\{S_1, \cdots, S_{n-2}\}$. So if S_{n-1} is one choice of matrix for the last generator, any other choice of matrix for the last generator is of the form $\gamma^{-1}S_{n-1}\gamma$. Furthermore, after changing basis so the matrices for the first $n-2$ generators are block diagonal, then by Schur's lemma every γ is block diagonal with scaled identity matrices as the blocks.

- If J is a form for the representation $\langle S_1, \cdots, S_{n-2}, S_{n-1}\rangle$, then $\gamma^* J\gamma$ is a form for the representation $\langle S_1, \cdots, S_{n-2}, \gamma^{-1}S_{n-2}\gamma\rangle$.

Example 4.3.2. *Consider the Young diagram* *, which has two 3-subdiagrams,* *, and* *. These subdiagrams correspond to the reduced Burau representation of B_3 and the trivial representation. Forming an induced representation on B_4 the first two generators can be given by*

$$S_1 = \left[\begin{array}{cc|c} q & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \quad \textit{and } S_2 = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ q & -q & 0 \\ \hline 0 & 0 & 1 \end{array}\right].$$

The matrices R_1 and R_2 are two different possibilities for the third generator. These matrices are conjugate by an element in the centralizer of $\langle S_1, S_2\rangle$ and ultimately give rise to different but equivalent representations of B_4.

$$R_1 = \begin{pmatrix} 1 & -\frac{q}{q^2+q+1} & 1 \\ 0 & \frac{(-q-1)q}{q^2+q+1} + 1 & q+1 \\ 0 & \frac{q\left(q^3+q^2+q+1\right)}{\left(q^2+q+1\right)^2} & -\frac{q^3}{q^2+q+1} \end{pmatrix} \qquad R_2 = \begin{pmatrix} 1 & -\frac{q}{q^2+q+1} & 3 \\ 0 & \frac{(-q-1)q}{q^2+q+1} + 1 & 3(q+1) \\ 0 & \frac{q\left(q^3+q^2+q+1\right)}{3\left(q^2+q+1\right)^2} & -\frac{q^3}{q^2+q+1} \end{pmatrix}$$

The matrix J_1 is the form for the representation $\langle S_1, S_2, R_1\rangle$ and the matrix J_2 is the form for the representation $\langle S_1, S_2, R_2\rangle$.

$$J_1 = \begin{pmatrix} \frac{q(q+1)^2(q^2+1)}{(q^2+q+1)^3} & -\frac{q(q+1)(q^2+1)}{(q^2+q+1)^3} & 0 \\ -\frac{q^2(q+1)(q^2+1)}{(q^2+q+1)^3} & \frac{q(q+1)^2(q^2+1)}{(q^2+q+1)^3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad J_2 = \begin{pmatrix} \frac{q(q+1)^2(q^2+1)}{(q^2+q+1)^3} & -\frac{q(q+1)(q^2+1)}{(q^2+q+1)^3} & 0 \\ -\frac{q^2(q+1)(q^2+1)}{(q^2+q+1)^3} & \frac{q(q+1)^2(q^2+1)}{(q^2+q+1)^3} & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

Example 4.3.3. *For the representation of B_6 from Example 4.3.1, $k = 1$ as λ only has one 5-subdiagram. In this case, the explicit matrices for the first 4 generators fully determined the last generator and the form.*

Chapter 5

BMW Representations

5.1 The BMW Algebras

The BMW algebras were discovered in the 1980's by Joan Birman and Hans Wenzl in [4], and simultaneously by Jun Murakami [25]. Following the notation of [4] and Zinno in [34], the BMW algebras $C_n(l, m)$ are a two parameter, l and m, family of algebras with $n-1$ generators. The invertible generators are denoted $G_1, \cdots, G_{n-1}$ which satisfy the following relations in terms of non-invertible elements denoted by E_i as follows:

$$G_i G_j = G_j G_i \quad \text{for } |i-j| > 1 \tag{5.1}$$

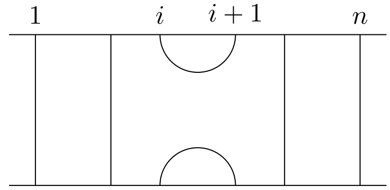
$$G_i G_{i+1} G_i = G_{i+1} G_i G_{i+1} \tag{5.2}$$

$$G_i^2 = m(G_i + l^{-1} E_i) - 1. \tag{5.3}$$

There are additional relations:

$$
\begin{aligned}
E_i E_{i\pm1} E_i &= E_i \\
G_{i\pm1} G_i E_{i\pm1} &= E_i G_{i\pm1} G_i = E_i E_{i\pm1} \\
G_{i\pm1} E_i G_{i\pm1} &= G_i^{-1} E_{i\pm1} G_i^{-1} \\
G_{i\pm1} E_i E_{i\pm1} &= G_i^{-1} E_{i\pm1} \\
E_{i\pm1} E_i G_{i\pm1} &= E_{i\pm1} G_i^{-1} \\
G_i E_i &= E_i G_i = l^{-1} E_i \\
E_i G_{i\pm1} E_i &= l E_i \\
E_i^2 &= (m^{-1}(l + l^{-1}) - 1) E_i
\end{aligned}
$$

The generating G_i's can be visualized by the usual braid diagram for σ_i as in Figure 1.2, and E_i can be visualized as the diagram in Figure 5.1. The relations are motivated by regular isotopy applied to the associated concatenated diagrams. This visualization is due to the fact that the BMW algebra is isomorphic to Kauffman's tangle algebra. One way to think of the BMW algebra is like a 2 parameter version of the Temperley–Lieb algebra with certain crossings allowed.



E_i

Figure 5.1: Braid-like element E_i.

Equations 5.1 and 5.2 show that $C_n(l, m)$ can be seen as a quotient of $\mathbb{C}[B_n]$ and there is a standard homomorphism sending $\sigma_i \mapsto G_i$. Also, $C_{n-1}(l, m) \subseteq C_n(l, m)$ and this respects the usual inclusion $B_{n-1} \subseteq B_n$. A representation ρ of the BMW algebra induces a representation of the braid group by mapping $\sigma_i \mapsto \rho(G_i)$.

Notice, if $E_i = 0$ equation 5.3 reduces to $G_i^2 = mG_i - 1$ which is very close to the defining relation for the Hecke algebras, equation 4.1 . The Hecke algebras are indeed isomorphic to a quotient of the BMW algebra best described by $E_i \mapsto 0$ and $G_i \mapsto lg_i$.

However, the Hecke algebra's are not equal to a subalgebra of the BMW because of an incompatibility of their multiplicative structures, see [34] for more detail. This isomorphic copy of the Hecke algebra inside $C_n(l, m)$ is typically denoted by H_n.

The dimension of $C_n(l, m)$ is $\frac{(2n)!}{2^n n!}$ which has been proved in many ways, but directly computed in [24] and [4]. BMW algebras are a deformation of the Brauer algebras in the same way that the Hecke algebras are a deformation of the complex algebra of the symmetric group. The Brauer algebras can be obtained from $C_n(l, m)$ by simple re-parameterization and specialization of $l = -i$, see Section 5 of [4].

5.2 The BMW Representations

Recall from Section 4.1 that the irreducible representations of the Hecke algebras are parameterized by the Young diagrams with the standard Young lattice describing the restriction rules in Figure 4.2. Analogously, the irreducible representations of the BMW algebras $C_n(l, m)$ are parameterized by a Bratteli diagram whose vertices are Young Diagram as show in Figure 5.2, but the restriction rule is quite different from the standard Young lattice. The new restriction rule is:

***BMW restriction rule**: A Young diagram λ_n in row n is connected to a Young diagram λ_{n+1} in row $n+1$ if λ_{n+1} is obtained from λ_n by adding or removing a single box.*

As depicted in Figure 5.2, the standard Young lattice occurs in the Bratteli diagram and corresponds to H_n the subalgebra of $C_n(l, m)$ isomorphic to the Hecke algebras. The induced representations of the braid group coming from the subalgebra H_n are the Jones representations, [34].

Notice that any λ_{n+1} in the $n+1$'st row is completely determined by the the set of diagrams in the n'th row connecting to λ_{n+1}. We can describe this property by saying

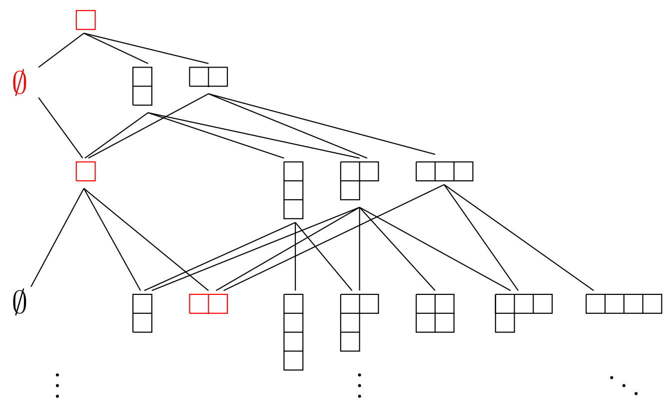


Figure 5.2: Bratteli diagrams for the restriction rules of the irreducible representations of the BMW algebras. The Lawrence-Krammer representation is shown in red.

a diagram is determined up to equivalence by the restriction rule (equivalent as in the sense of Definition 4.1.4 from Section 4.1).

Similar to how the the Burau representation is one irreducible summand of the Jones representations, the Lawrence-Krammer representation is one summand of the BMW representations, colored red in Figure 5.2. The Lawrence-Krammer representation was first discovered by Ruth Lawrence [19] and famously studied by Daan Krammer and Stephen Bigelow. It was originally defined as a homological representations coming from a mapping class group action, but was proved later by Zinno [34] to come from the BMW algebras.

5.3 Explicit Matrices for the BMW Representations

To compute the three 1-dimensional BMW representations of B_2, first we choose a basis for the algebra as a vector space over $C(l, m)$, for example $\{1, G, G^{-1}\}$. Then we let the G act on this ordered basis to get a 3×3 representation of B_2. This left regular representation is a direct sum of thee 1-dimensional representations, shown in the diagonalization.

$$\begin{pmatrix} 0 & -\frac{m}{l}-1 & 1 \\ 1 & m+\frac{1}{l} & 0 \\ 0 & \frac{1}{l} & 0 \end{pmatrix} \sim \begin{pmatrix} \frac{1}{l} & 0 & 0 \\ 0 & \frac{1}{2}(m-1\sqrt{-4+m^2}) & 0 \\ 0 & 0 & \frac{1}{2}(m+\sqrt{-4+m^2}) \end{pmatrix}$$

So the three 1-dimensional representations are $\varphi_1(G) = \frac{1}{l}$, $\rho_1(G) = \frac{1}{2}(m-1\sqrt{-4+m^2})$ and $\rho_2(G) = \frac{1}{2}(m+1\sqrt{-4+m^2})$.

$$\rho_i(E) = \frac{1}{m}(\rho_i(G) - \rho_i(G)^{-1}) - 1 = 0$$

Which shows that both ρ_i's are actually representations of the Hecke algebra so they correspond to the two Young diagrams in row 2 of the Bratteli diagrams. The third representation φ corresponds to the $\emptyset$ diagram.

Example 5.3.1. *In [4], Birman and Wenzl computed the representation of B_3 corresponding to the single box Young diagram.*

$$\sigma_i \mapsto \begin{pmatrix} l^{-1} & m & 0 \\ 0 & m & 1 \\ 0 & -1 & 0 \end{pmatrix}, \sigma_2 \mapsto \begin{pmatrix} 0 & 0 & -1 \\ 0 & l^{-1} & l^{-1}m \\ 1 & 0 & m \end{pmatrix}$$

5.4 Sesquilinearity

Using Morse theoretic arguments, Budney proved that the Lawrence Krammer is sesquilinear [7]. In this section, we will extend Budney's result to all of the BMW representations.

Theorem 5.4.1. *If ρ is an irreducible BMW representation of B_n then there exists a non-degenerate, sesquilinear matrix J so that for all M in the image of ρ, $M^*JM = J$.*

To make sense of the $*$ operation in Theorem 5.4.1, we need to define an involution of the coefficients. The relevant involution for the BMW algebra is $l \mapsto \frac{1}{l}$, $m \mapsto m$, and $\alpha \mapsto \frac{1}{\alpha}$ where $m = \alpha + \frac{1}{\alpha}$. We will denote this involution by ϕ. So to show sesquilineararity in this context is to show that the representations are equivalent to their ϕ-twisted contragrediant representation using ϕ to define $*$.

The proof of Theorem 5.4.1 is exactly analogous to the proof of Theorem 4.0.1 showing that the Jones representations are sesquilinear, excluding the positive definite argument. It is only necessary to prove the following Lemma.

Lemma 5.4.2. *If ρ is an irreducible BMW representation of B_n, then ρ is equivalent to its ϕ-twisted contragredient representation.*

Proof. Analogously to Lemma 4.2.3, we will prove this result by induction on n.

Let ρ be an irreducible BMW representation of B_2. As shown in the Section 5.3, there are three possible 1-dimensional representations given by $\varphi_1(G) = \frac{1}{l}$, $\rho_1(G) = \frac{1}{2}(m - 1\sqrt{-4+m^2})$ and $\rho_2(G) = \frac{1}{2}(m + 1\sqrt{-4+m^2})$.

In the diagonalization process to compute these representations, we introduced a square-root term $\sqrt{-4+m^2}$. Extending ϕ to the field including this term, we define $\phi(\sqrt{-4+m^2}) = -\sqrt{-4+m^2}$. Each of these one-dimensional representations are equal to thier ϕ-twisted contragredient representation.

Inductively moving forward, let ρ be an irreducible BMW representation of B_n, and $\tilde{\rho}$ be the ϕ-twisted contragadient of ρ. Let ρ correspond to Young diagram λ_1 and $\tilde{\rho}$ correspond to λ_2. As described in Section 5.2, λ_1 and λ_2 are completely determined by the BMW restriction rule. The inductive step is the exactly the same as in Lemma 4.2.3.

□

5.4.1 Positive-Definiteness

One motivation for showing the BMW representations are sesquilinear is to find discrete specializations of the parameters. Theorem 3.3.3 proved that we can find infinitely many discrete specializations of a parameterized representation, given certain requirements about the positive definiteness of the sesquilinear form J. This section discusses the positive definiteness of the forms for the BMW representations.

Conjecture: *The forms for the BMW representations, found in Theorem 5.4.1, are positive definite for specializations in some open neighborhood in* $\mathbb{C}^2$.

This conjecture has been experimentally verified for several of the smaller indexed BMW representations as described in Examples 5.4.3 and 5.4.4 to follow, but has not been proven in general. Since the proof of Theorem 5.4.1 was completely analogous to that of Theorem 4.0.1, at first glance there is hope to repeat the positive definiteness argument that worked for the Jones representations. However, there is a major obstacle that prevents this approach from generalizing to the BMW representations. The forms for the Jones representations found in Theorem 4.0.1 were proven to be positive definite in a complex neighborhood of 1 by using the fact that the Hecke algebras are a deformation of the complex symmetric algebras. That is at $q = 1$, $H_q(n)$ collapses to $\mathbb{C}[\Sigma_n]$. Since Σ_n is a finite group, its representations are unitary. Now in a similar way, the BMW algebras are a deformation of the Brauer algebras. However it is unknown whether the irreducible representations of the Brauer algebras are unitary/sesquilinear or not. So some further investigation into the representation theory of the Brauer algebras could lead to a general proof the conjecture.

Example 5.4.3.

One of the BMW representation of B_4 is given by:

$$\sigma_1 \mapsto \begin{bmatrix} \frac{1}{a} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \frac{1}{L} & \frac{a+\frac{1}{a}}{L} & \frac{a\left(a+\frac{1}{a}\right)}{L} \\ 0 & 1 & 0 & 0 & a+\frac{1}{a} & 0 \\ 0 & 0 & 1 & 0 & 0 & a+\frac{1}{a} \end{bmatrix} \qquad \sigma_2 \mapsto \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \frac{1}{L} & \frac{a+\frac{1}{a}}{L} & a+\frac{1}{a} & 0 & 0 \\ 1 & 0 & a+\frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & a+\frac{1}{a} & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{a} \end{bmatrix}$$

$$\sigma_3 \mapsto \begin{bmatrix} \frac{1}{L} & a+\frac{1}{a} & 0 & 0 & \frac{a+\frac{1}{a}}{a} & 0 \\ 0 & a+\frac{1}{a} & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 0 & 0 & 0 & a+\frac{1}{a} & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

J can be computed as the diagonal matrix with the following entries on the diagonal

$$\begin{aligned} J_{1,1} =& 2 \\ J_{2,2} =& -\frac{2a\left(L^2+1\right)\left(2a^2L-aL^2-a+2L\right)}{(a-L)^2(aL-1)^2} \\ J_{3,3} =& \frac{2\left(L^2+1\right)\left(a^3+L\right)\left(a^3L+1\right)}{a(a-L)(aL-1)\left(2a^2L-aL^2-a+2L\right)} \\ J_{4,4} =& \frac{2(a+L)\left(a^5L^2+a^4L-a^3L^2-a^3+a^2L^3+a^2L-aL^2-L\right)}{a\left(L^2+1\right)\left(a^3+L\right)(aL-1)} \\ J_{5,5} =& \frac{2(a+L)(aL+1)\left(a^3L+1\right)\left(2a^3L^2+a^3+a^2L+aL^2+L^3+2L\right)}{a\left(L^2+1\right)(aL-1)\left(a^5L^2+a^4L-a^3L^2-a^3+a^2L^3+a^2L-aL^2-L\right)} \\ J_{6,6} =& -\frac{2\left(a^5-L\right)(a+L)(aL+1)\left(a^3L+1\right)}{a^3(aL-1)\left(2a^3L^2+a^3+a^2L+aL^2+L^3+2L\right)} \end{aligned}$$

A short computation shows that $J(a,l) = J(i,1) = 2Id$, giving a point where J is positive definite. Continuity of the determinant implies that J is positive definite in a neighborhood of $(i,1)$ on the complex torus. It is difficult to determine explicitly the radius of this neighborhood. However, one can choose a Salem number and through trial and error, find powers of the Salem number that are very close to $(i,1)$ where the form stays positive definite. Taking the Salem number $S = \frac{1}{2} + \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{-1+2\sqrt{2}}}$, specializing $a = S^{15}$ and $L = S^3$ leaves J positive definite at the complex places of $\mathcal{O}_{\mathbb{Q}(S)}$. So this representation is discrete at the specialization $a = S^{15}$, $L = S^3$.

Example 5.4.4.

Taking $m = 0$ in the BMW algebra yields a quotient of the $\mathbb{C}[B_n]$ where each generator $G_i^2 = -1$, or rather each generator has order 4. It is known that this quotient of the braid group yields a finite group exactly when $n = 3$ [18, pg 81]. So with this information, we know there is a neighborhood of $m = 0$ where all of the BMW representations of B_3 have a positive definite form.

Chapter 6

Lattices and Commensurability

6.1 Commensurability

The irreducible Jones representations corresponding to rectangular Young diagrams, as in Example 4.3.1, are particularly interesting and lead to some further questioning. What is special about these representations is that they each have only one $(n-1)$-subdiagram. This implies that both the restriction representation and the full representation are irreducible, of the same dimension and use the same form J. Both representations map into the same unitary group. How can we mimic this situation for the other irreducible Jones representations for the non-rectangular diagrams? Can we get two representations to map into the "same" unitary group? The answer is yes! The approach is to fix a representation and specialize to two different powers of the same Salem number. The ring of integers $\mathcal{O}_K$ will stay the same, but the defining sesquilinear forms might be very different.

Recall the notation of K, L, $\mathcal{O}_K$ and ϕ from Section 3.3. In general, fixing a number ring $\mathcal{O}_K$ and dimension m, the group $U_m(J, \phi, \mathcal{O}_K)$ is determined by the form J. Notice that $U_m(J, \phi, \mathcal{O}_K) = U_m(\lambda J, \phi, \mathcal{O}_K)$ for every $\lambda \in L$, and that the form J is not completely unique. This motivates that following definition.

Definition 6.1.1. *Matrices J and H are equivalent over K if $Q^*JQ = \lambda H$ for some $Q \in GL_m(K)$ and $\lambda \in Fix(\phi)$.*

It would be nice if equivalent forms gave rise to *equal* unitary groups, but this is not true in general. However, in the careful scenario that the unitary group is a lattice in $SL(\mathbb{R})$, then changing the form by equivalence yields "the same" lattice, up to commensurability in the following sense.

Definition 6.1.2. *Two groups G_1 and G_2 are **commensurable** if there are finite index subgroups $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ so that H_1 is isomorphic to H_2.*

Definition 6.1.3. *A **lattice** in a semisimple Lie group G is a discrete subgroup of G with finite covolume.*

For our purposes, we will take $G = SL_m(\mathbb{R})$ or $PSL_m(\mathbb{R})$.

Proposition 6.1.4. *Assume $SU_m(J_1, \phi, \mathcal{O}_K)$ and $SU_m(J_2, \phi, \mathcal{O}_K)$ are lattices in $SL_m(\mathbb{R})$. If J_1 and J_2 are equivalent over K, then $SU_m(J_1, \phi, \mathcal{O}_K)$ is commensurable to $SU_m(J_2, \phi, \mathcal{O}_K)$*

Proof. Let $\lambda J_1 = Q^* J_2 Q$ for some $Q \in GL_m(K)$ and $\lambda \in Fix(\phi)$. For notational clarity, denote $SU(J_i, \mathcal{O}_K) = SU_m(J_i, \phi, \mathcal{O}_K)$.

Since scalar multiplication commutes with matrix multiplication, then $M^*JM = J$ if and only if $M^*\lambda JM = \lambda J$. So scaling the form preserves the unitary group, and with out loss of generality we may assume $\lambda = 1$. It is easy to see that $M^*JM = J$ if and only if $(Q^*M^*Q^{*-1})(Q^*JQ)(Q^{-1}MQ) = Q^*JQ$, which seems like it implies that $SU(Q^*J_1Q, \mathcal{O}_K) = Q^{-1}SU(J_1, \mathcal{O}_K)Q$. However, since Q has coefficients in K, $Q^{-1}MQ$ may not have coefficients in $\mathcal{O}_K$, so we can only conclude that $Q^{-1}SU(J, \mathcal{O}_K)Q \subseteq SU(Q^*JQ, K)$. To avoid this, we need to pass to a finite index subgroup.

Since K is the ring of fractions of $\mathcal{O}_K$, then there exists $\gamma \in \mathcal{O}_K$ so that $\gamma Q \in M_m(\mathcal{O}_K)$. As a ring of integers of an algebraic extension, $\mathcal{O}_K$ is a Dedekind domain and every quotient is finite. So $\mathcal{O}_K/\langle\gamma^2\rangle$ is finite and $SU(J_1, \mathcal{O}_K/\langle\gamma^2\rangle)$ is finite. The kernel N of the quotient map $SU(J_1, \mathcal{O}_K) \to SU(J_1, \mathcal{O}_K/\langle\gamma^2\rangle)$ has finite index in $SU(J_1, \mathcal{O}_K)$.

Any element B in the kernel has the form $B = Id + \gamma^2 A$ for some matrix $A \in M_m(\mathcal{O}_K)$. Inserting $Q^* J_2 Q$ for J_1 into the equation $B^* J_1 B = J_1$, gives that that QBQ^{-1} fixes the form J_2. Because Q has coefficients over K, QBQ^{-1} has coefficients in K and not necessarily in $\mathcal{O}_K$. However, since $QBQ^{-1} = Id + (\gamma Q)A(\gamma Q^{-1})$, and both A and γQ are integral, then QBQ^{-1} is also integral. Thus $QBQ^{-1} \in SU(J_2, \mathcal{O}_K)$.

Since $SU(J_1, \mathcal{O}_K)$ is a lattice, and N is a finite index subgroup, then N is also a lattice in $SL(\mathbb{R})$ with finite covolume. Thus QNQ^{-1} has finite covolume in $SL(\mathbb{R})$ and is therefore a lattice in G. So QNQ^{-1} is a sublattice of $SU(J_2, \mathcal{O}_K)$ and must have finite index.

This shows that N is a finite index subgroup $SU(J_1, \mathcal{O}_K)$ and QNQ^{-1} is finite index in $SU(J_2, \mathcal{O}_K)$. □

So how does this lattice information apply to the Jones representations? Firstly, after a rescaling and reparameterization, the Jones representations can be made to have determinant ± 1, allowing the image to land in an $PSU(J, \phi, \mathcal{O}_K)$ instead of just $U(J, \phi, \mathcal{O}_K)$. Secondly, an arithmetic group theory result of Harish-Chandra, that is formalized in our setting in Chapter 6 of Witte [23], states that $SU_m(J, \phi, \mathcal{O}_K)$ is a lattice in $SL_m(\mathbb{R})$ under the exact Salem number circumstances as required by Theorem 3.3.3. So we can restate Corollary 4.2.5 using this new vocabulary.

Corollary 6.1.5. *For each irreducible Jones representation, after a change of parameter, there are infinitely many Salem numbers s so that specializing q to a powers of s maps the braid group into a lattice in $PSL_m(\mathbb{R})$.*

Proof. Let ρ_q be an irreducible Jones representation of dimension m. The images of the braid generators under ρ_q have determinant $\pm q^k$ for some $k \in \mathbb{N}$. After a change of variable $q = y^m$ and scaling the generators by $\frac{1}{y^{m-k}}$, this adjusted representation $\tilde{\rho}_y$ maps into $PSU_m(J^y, \mathbb{Z}[y^{\pm 1}])$.

The subgroup B_n^2 of squared braids is a non-central normal subgroup of B_n of finite index. The restriction $\tilde{\rho}_y|$ maps B_n^2 into $SU_m(J_y, \mathbb{Z}[y^{\pm 1}])$, and by Theorem 3.3.3, there exists infinitely many Salem numbers s so that the specialization $\rho_s|$ at $y = s$ is discrete. Further by the the result in [23], these specializations make $SU_m(J_s, \mathcal{O}_K)$ lattices in $SL_m(\mathbb{R})$. Finite index arguments imply $PSU_m(J_s, \mathcal{O}_K)$ is a lattice in $PSL_m(\mathbb{R})$.

□

Since our goal is to obtain commensurable lattices as images of our Jones representations, and it is more natural to think of lattices in $SL_m(\mathbb{R})$ instead of in $PSL_m(\mathbb{R})$, we may simply pass to the finite index subgroup B_n^2 and continue to think only about lattices in $SL_m(\mathbb{R})$.

So now we want to be able to apply Proposition 6.1.4 to understand commensurability classes of these lattices, and hence understand the equivalence classes of the defining forms. In general, it is difficult to determine when two forms are equivalent. The following theorem gives a complete classification of the sesquilinear forms in a very specific algebraic setting that applies to the Salem number field scenario.

Theorem 6.1.6 (Scharlau [27], Ch.10)**.** *If L is a global field and $K = L(\sqrt{\delta})$, sesquilinear forms over K/L are classified by dimension, determinant class and the signatures for those orderings of L for which δ is negative.*

This classification relies on the *determinant class* which we now define. Recall, for a Salem number s, we get the following tower of fields.

$$\begin{array}{ccccc} \mathbb{Q}(s) = K & \qquad & K^\times & & \\ \Big|\, 2 & & \Big\downarrow Norm & & \\ \mathbb{Q}(s + \frac{1}{s}) = L & & L^\times & \qquad & (L^\times)^2 \subseteq Norm(K^\times) \\ \Big| & & & & \\ \mathbb{Q} & & & & \end{array}$$

The Galois group of K/L is generated by ϕ which maps $s \mapsto \frac{1}{s}$. There is a multiplicative group homomorphism $Norm : K^\times \to L^\times$ given by $Norm(\alpha) = \alpha\alpha^\phi$, where $K^\times = K - \{0\}$. Notice for $\beta \in L$, $Norm(\beta) = \beta\beta^\phi = \beta^2$. So $(L^\times)^2 \subseteq Norm(K)$.

Definition 6.1.7. *The **determinant class** of a sesquilinear form H over K/L is the coset of* $\det(H)$ *in* $K^\times/Norm(K^\times)$,

$$[\det(H)] = \det(H)Norm(K).$$

Taking $\delta = (s - \frac{1}{s})^2$, K can be rewritten as $K = L(\sqrt{\delta})$. Thus we can restate Scharlau's classification in the specific context of Salem numbers.

Theorem 6.1.8 (Scharlau restated)**.** *Sesquilinear forms over K/L are classified by dimension, determinant class and the signatures for those orderings of L for which $(s-\frac{1}{s})^2$ is negative.*

In odd dimension, it is very simple to show that *all* sesquilinear forms have the same determinant class, up to scaling. However, for even dimension, the situation is very unclear.

Proposition 6.1.9. *For every odd dimensional invertible sesquilinear matrices H and J over K,* $[\det(H)] = [\det(\lambda J)]$ *for $\lambda \in L$.*

Proof. Let H and J be sesquilinear matrices over K of dimension $2k+1$. Hermitian guarantees both H and J are diagonalizable with diagonal entrees fixed by ϕ. So, the determinant of both H and J are elements in L. Let d_H and d_J denote the nonzero determinants of both H and J. Thus

$$d_H = \frac{d_H}{d_J}d_J \overset{mod (L^\times)^2}{\equiv} (\frac{d_H}{d_J})^{2k+1}d_J = \det(\frac{d_H}{d_J}J).$$

Since $(L^\times)^2 \subseteq Norm(K)$, then H and λJ have the same determinant class for $\lambda = \frac{d_H}{d_J} \in L$. □

As a result, to determine whether two forms of the same odd dimension are equivalent, it suffices only to do check they have the same signatures.

Theorem 6.1.10. *For J_t a sesquilinear form that is positive definite for t in a neighborhood η of 1, there are infinitely many Salem numbers s and integers n, m, so that in all* *odd dimension, $SU_{2k+1}(J_{s^n}, \phi, \mathcal{O}_K)$ and $SU_{2k+1}(J_{s^m}, \phi, \mathcal{O}_K)$ are commensurable, discrete groups.*

Proof. By Lemma 3.3.2 there are infinitely many Salem numbers s and integers n, m so that every complex Galois conjugate of s^m and s^n lie in η. Fix one such Salem number s, and K, L, and δ as above.

By Theorem 6.1.6, sesquilinear forms are completely classified by dimension, determinant class, and the signatures for the places of L for which $(s - \frac{1}{s})^2$ is negative. By Proposition 6.1.9, J_{s^n} and λJ_{s^m} have the same determinant class for λ in L, namely $\lambda = \frac{\det J_{s^n}}{\det J_{s^m}}$.

Let σ be a complex placement of L. Then $\sigma(s^m)$ is a complex Galois conjugate of s^m, and similarly for $\sigma(s^n)$ and s^n. Since n and m were chosen so that all of the complex Galois conjugate of s^m and s^n have arguments in η, then $J_{\sigma(s^m)}$ and $J_{\sigma(s^n)}$ are positive definite. Moreover, $\frac{\det J_{s^n}}{\det J_{s^m}}$ and $\sigma(\frac{\det J_{s^n}}{\det J_{s^m}})$ are both positive, making $\lambda > 0$. So regardless of whether $\sigma((s - \frac{1}{s})^2)$ is positive or negative, the forms $J_{\sigma(s^i)}$ have the same signature.

Thus, J_{s^n} is equivalent to λJ_{s^m}, and $SU(J_{s^n}, \phi, \mathcal{O}_K)$ is commensurable to $SU(J_{s^m}, \phi, \mathcal{O}_K)$. The groups are discrete by Theorem 3.2.3.

□

Corollary 6.1.11. *Let $\rho_t : G \to SL_{2k+1}(\mathbb{Z}[t, t^{-t}])$ be a group representation with a parameter t. Suppose there exists a matrix J_t so that:*

1. *for all M in the image of ρ_t, $M^* J_t M = J_t$, where $M^*(t) = M^\intercal(\frac{1}{t})$,*
2. *$J_t = (J_{\frac{1}{t}})^\intercal$,*
3. *J_t is positive definite for t in an neighborhood η of 1*

Then, there exists infinitely many Salem numbers s, so that for infinitely many integers n, m the specializations ρ_{s^m} at $t = s^m$ and ρ_{s^n} at $t = s^n$ map into commensurable lattices of $SL_{2k+1}(\mathbb{R})$.

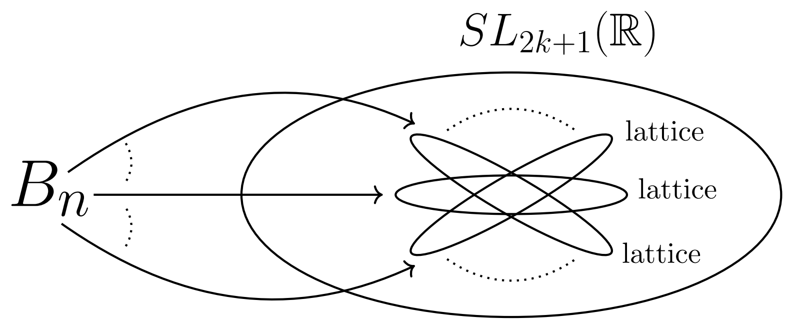


Figure 6.1

Example 6.1.12. *The reduced Burau representation of B_4 is 3 dimensional and, after the appropriate rescaling to have determinant 1, satisfies Corollary 6.1.11. So certain powers of the specializations in Example 3.4.2 map into commensurable lattices in $SL_3(\mathbb{R})$.*

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