PROOF OF THE FOLKLORE OC RELATION IDENTITY

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ABSTRACT. One presentation for the welded braid groups is a quotient for of the virtual braid groups by the *over crossing commute*, or OC relation of the form $\tau_i\sigma_{i+1}\tau_i = \sigma_{i+1}\sigma_i\tau_{i+1}$. The OC relation is often written in a different form $\sigma_{i,k}\sigma_{i,j} = \sigma_{i,j}\sigma_{i,k}$. It is known to experts that these relations are equivalent and we provide a short proof here.

The virtual braid group on n strands, vB_n, has a presentation generated by $\sigma_1, \ldots \sigma_{n-1}$ and $\tau_1, \ldots, \tau_{n-1}$ with the following relations:

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$ (Far Commutativity)
(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i \leq n-2$ (Braid Relation)
(3) $\tau_i^2 = 1$ for $1 \leq i \leq n-1$ (τ is a Transposition)
(4) $\tau_i\tau_j = \tau_j\tau_i$ for $|i-j| > 1$ (τ Far Commutativity)
(5) $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$ for $1 \leq i \leq n-2$ (τ Braid Relation)
(6) $\sigma_i\tau_j = \tau_j\sigma_i$ for $|i-j| > 1$ (Mixed Far Commutativity)
(7) $\tau_{i+1}\sigma_i\tau_{i+1} = \tau_i\sigma_{i+1}\tau_i$ for $1 \leq i \leq n-2$ (Mixed Braid Relation)

The welded braid group wB_n is a quotient of vB_n by one additional relation called the Over Crossings Commute relation, or "OC" relation, of the form $\tau_i\sigma_{i+1}\tau_i = \sigma_{i+1}\sigma_i\tau_{i+1}$.

There are several important elements in vB_n (and in wBn) called $\sigma_{i,j}$ which are of the form

$$\tau_i\tau_{i+1}\ldots\tau_{j-2}\tau_{j-1}\sigma_{j-1}\tau_{j-2}\ldots\tau_{i+1}\tau_i$$

when $i < j$ and

$$\tau_{i-1}\tau_{i-2}\ldots\tau_{j-2}\tau_{j-1}\sigma_j\tau_j\tau_{j-1}\ldots\tau_{i-1}$$

when $j < i$.

Theorem. *In the welded braid group, the following relation holds for all i, j, k,*

$$\sigma_{i,k}\sigma_{i,j} = \sigma_{i,j}\sigma_{i,k}.$$

Proof. In this proof, we will show that $\sigma_{ik}\sigma_{ij} = \sigma_{ij}\sigma_{ik}$ in the case where $i < j, k$ as the other case is analagous.

We begin by fixing the coordinate i, and without loss of generality, let $j < k$. We prove, by induction, that $\sigma_{i,i+1}\sigma_{i,j} = \sigma_{i,j}\sigma_{i,i+1}$ (the case that $\sigma_{i,i+k}\sigma_{i,j} = \sigma_{i,j}\sigma_{i,i+k}$, where $i + k < j$ is proven similarly). For the base case, notice that

$$\begin{aligned}
\sigma_{i,i+1}\sigma_{i,i+2}\sigma_{i,i+1}^{-1}\sigma_{i,i+2}^{-1} &= (\tau_i\sigma_i)(\tau_i\tau_{i+1}\sigma_{i+1}\tau_i)(\sigma_i^{-1}\tau_i)(\tau_i\sigma_{i+1}^{-1}\tau_{i+1}\tau_i) \\
&= \tau_i\sigma_i\tau_i\tau_{i+1}\tau_i\tau_i\sigma_{i+1}\tau_i\sigma_i^{-1}\sigma_{i+1}^{-1}\tau_{i+1}\tau_i \\
&= \tau_i\sigma_i\tau_i\tau_{i+1}\tau_i\tau_{i+1}\sigma_i\tau_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}\tau_{i+1}\tau_i \\
&= \tau_i\sigma_i\tau_{i+1}\tau_i\sigma_i\tau_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}\tau_{i+1}\tau_i \\
&= \tau_i\tau_{i+1}\tau_i\sigma_{i+1}\sigma_i\tau_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}\tau_{i+1}\tau_i \text{(By OC)} \\
&= \tau_i\tau_{i+1}\tau_i\tau_i\sigma_{i+1}\sigma_i\sigma_i^{-1}\sigma_{i+1}^{-1}\tau_{i+1}\tau_i \text{(By OC)} \\
&= id
\end{aligned} \tag{1}$$

Now we suppose that $\sigma_{i,i+1}\sigma_{i,j-1} = \sigma_{i,j-1}\sigma_{i,i+1}$ and show that $\sigma_{i,i+1}\sigma_{i,j} = \sigma_{i,j}\sigma_{i,i+1}$.

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\begin{aligned}
\sigma_{i,i+1}\sigma_{i,i+2}\sigma_{i,i+1}^{-1}\sigma_{i,i+2}^{-1} &= (\tau_i\sigma_i)(\tau_i\tau_{i+1}\ldots\tau_{j-2}\tau_{j-1}\sigma_{j-1}\tau_{j-2}\tau_{j-3}\ldots\tau_{i+1}\tau_i)\cdot \\
&\qquad \cdot(\sigma_i^{-1}\tau_i)(\tau_i\tau_{i+1}\ldots\tau_{j-3}\tau_{j-2}\sigma_{j-1}^{-1}\tau_{j-1}\tau_{j-2}\ldots\tau_{i+1}\tau_i) \\
&= \tau_i\sigma_i\tau_i\tau_{i+1}\ldots\tau_{j-2}\tau_{j-1}\tau_{j-2}\tau_{j-2}\sigma_{j-1}\tau_{j-2}\tau_{j-3}\ldots\tau_{i+1}\tau_i\cdot \\
&\qquad \cdot\sigma_i^{-1}\tau_{i+1}\ldots\tau_{j-3}\tau_{j-2}\sigma_{j-1}^{-1}\tau_{j-2}\tau_{j-2}\tau_{j-1}\tau_{j-2}\ldots\tau_{i+1}\tau_i \\
&= \tau_i\sigma_i\tau_i\tau_{i+1}\ldots\tau_{j-2}\tau_{j-1}\tau_{j-2}\tau_{j-1}\sigma_{j-2}\tau_{j-1}\tau_{j-3}\ldots\tau_{i+1}\tau_i\cdot \\
&\qquad \cdot\sigma_i^{-1}\tau_{i+1}\ldots\tau_{j-3}\tau_{j-1}\sigma_{j-2}^{-1}\tau_{j-1}\tau_{j-2}\tau_{j-1}\tau_{j-2}\ldots\tau_{i+1}\tau_i \\
&= \tau_i\sigma_i\tau_i\tau_{i+1}\ldots\tau_{j-2}\tau_{j-1}\tau_{j-2}\tau_{j-1}\sigma_{j-2}\tau_{j-3}\ldots\tau_{i+1}\tau_i\cdot \\
&\qquad \cdot\sigma_i^{-1}\tau_{i+1}\ldots\tau_{j-3}\sigma_{j-2}^{-1}\tau_{j-1}\tau_{j-2}\tau_{j-1}\tau_{j-2}\ldots\tau_{i+1}\tau_i \\
&\qquad\qquad \text{(by Far Commutivity)} \\
&= \tau_i\sigma_i\tau_i\tau_{i+1}\ldots\tau_{j-2}\tau_{j-2}\tau_{j-1}\tau_{j-2}\sigma_{j-2}\tau_{j-3}\ldots\tau_{i+1}\tau_i\cdot \\
&\qquad \cdot\sigma_i^{-1}\tau_{i+1}\ldots\tau_{j-3}\sigma_{j-2}^{-1}\tau_{j-2}\tau_{j-1}\tau_{j-2}\tau_{j-2}\ldots\tau_{i+1}\tau_i \\
&\qquad\qquad \text{(by } \tau \text{ Braid Relation)} \\
&= \tau_i\sigma_i\tau_i\tau_{i+1}\ldots\tau_{j-1}\tau_{j-2}\sigma_{j-2}\tau_{j-3}\ldots\tau_{i+1}\tau_i\cdot \\
&\qquad \cdot\sigma_i^{-1}\tau_{i+1}\ldots\tau_{j-3}\sigma_{j-2}^{-1}\tau_{j-2}\tau_{j-1}\ldots\tau_{i+1}\tau_i \\
&\qquad\qquad \text{(by } \tau \text{ Braid Relation)} \\
&= \sigma_{i,i+1}\sigma_{i,j-1} = \sigma_{i,j-1}\sigma_{i,i+1} = id.
\end{aligned} \tag{2}
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The other cases on i, j, k follow similarly. □