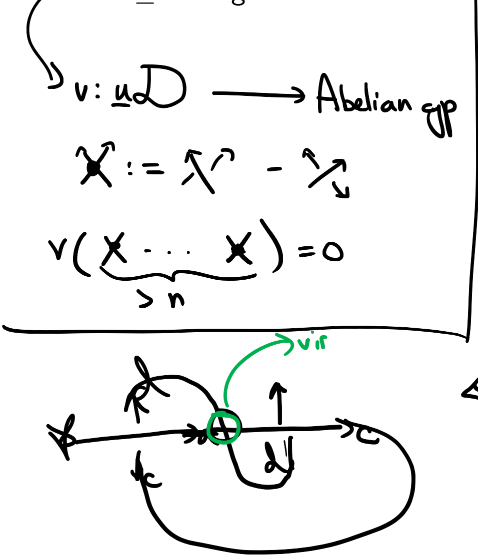
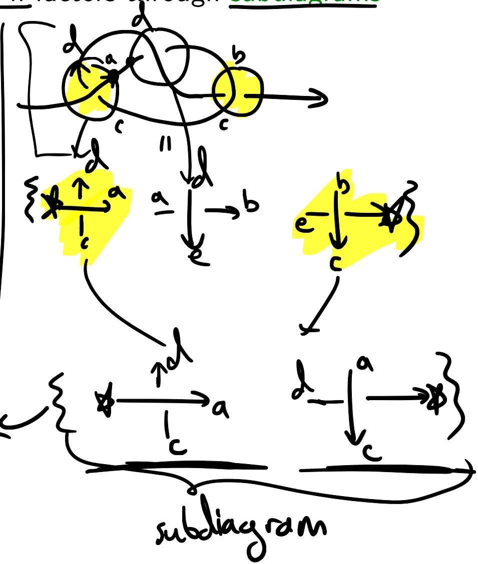
The Goussarov-Polyak-Viro (GPV) Theorem

July 21, 2021

“Statement of the theorem”

“A finite type invariant of degree n factors through subdiagrams with $\leq n$ Xings”

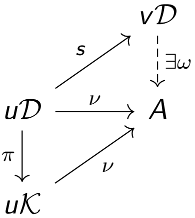




Statement of the theorem

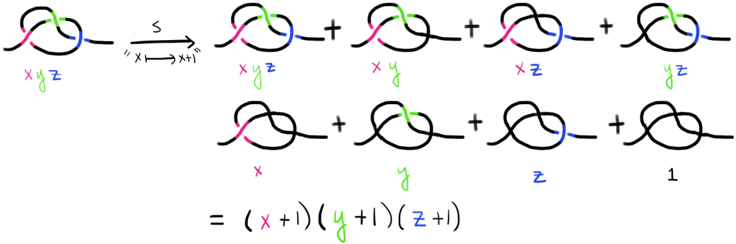
Theorem (Goussarov, Polyak, Viro): Given a finite type invariant $\nu : u\mathcal{D} \to A$ of degree n, there exists $\omega : v\mathcal{D} \to A$ such that:

1. $\omega \circ s = \nu$ (the following diagram commutes)
2. $\omega = 0$ on diagrams with $> n$ (real or double point) Xings.

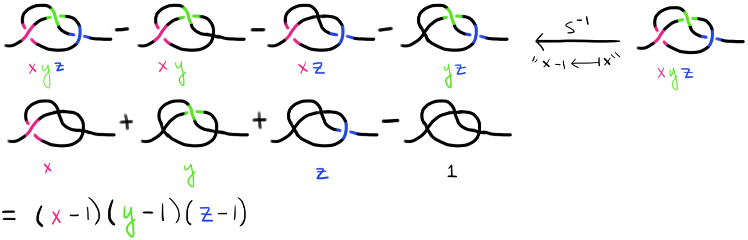


The map s and its inverse

The map $s : D \mapsto \sum_{D' \subseteq D} D'$ can be thought of as "$x \mapsto x + 1$":



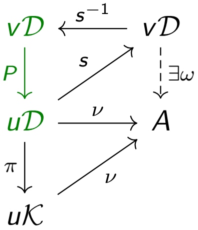
so its inverse is $s^{-1} : D \mapsto \sum_{D' \subseteq D} (-1)^{|D' - D|} D'$, or "$x \mapsto x - 1$":



The map P: properties

We want a map $P : v\mathcal{D} \to u\mathcal{D}$ which satisfies:

1. $\nu \circ P = \nu$ on real knot diagrams
2. $\nu \circ P \circ s^{-1} = 0$ on diagrams with $> n$ (real or singular) Xings

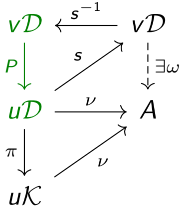


Then we can define

$$\boxed{\omega = \nu \circ P \circ s^{-1}}$$

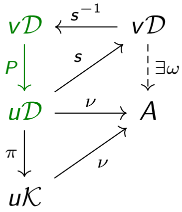
The map P: what it actually "is" (Roukema's version)

1. Make a first-come-first-serve tree



The map P: what it actually "is" (Roukema's version)

1. Make a first-come-first-serve tree
2. Change all bad Xings to good Xings (using the double point relation)



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1. Make a first-come-first-serve tree
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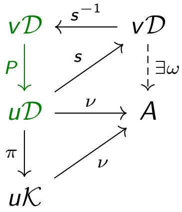
3. Sweep the first real crossing on the double point tree



The map P: what it actually "is" (Roukema's version)

1. Make a first-come-first-serve tree
2. Change all bad Xings to good Xings (using the double point relation)
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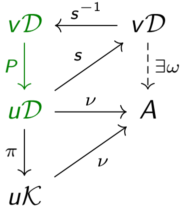
4. Repeat steps 1, 2 and 3 until...



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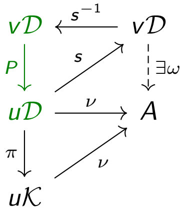
5. Set to 0 if $> n$ double points, otherwise, connect strands off the tree in a good way to make a knot



The map P: why it works

To show that P works, we must show:

1. $\nu \circ P = \nu$ on real knot diagrams
2. $\nu \circ P \circ s^{-1} = 0$ on $> n$ Xings

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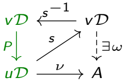

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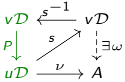

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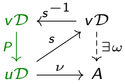

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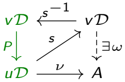

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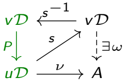

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So we are done! But...

Confusions

- Is it necessary to repeat step 1?
- How important is a first-come-first-serve tree? Can we build other trees?
- How can we change the ordering of the issues?
- Is there a more "direct" description of ω?

Generalizations

- To (not-long) knots, tangles, links?